Directorate of Distance Education

UNIVERSITY OF JAMMU
JAMMU

STUDY MATERIAL

For

B.A. 2ND SEMESTER

SUBJECT : MATHEMATICS **UNIT: I-V**

COURSE NO. : MA - 201 **LESSON NO. 1-20**

Course Co-ordinator
Dr. Rajber Singh Sodhi

Proof Reading & Content Reading by
Dr. Tirth Ram

http:/www.distanceeducationju.in

Printed & Published on behalf of the Directorate of Distance Education, University of Jammu by the Director, DDE, University of Jammu, Jammu-180006.

DESCRIPTIVE MATHEMATICS MA-201

• Printed by : Rohini Printers / 21 / 200

MATHEMATICS

Course No.: MA-201 **Title: Mathematics**

Duration of Exam. : 3 Hrs **Total Marks: 100**

Credit: 04 **Theory Examination: 80**

Internal Assessment: 20

UNIT: I

Linear and Bernoulli's differential equations. Exact and non-exact differential equation, Differential equations solvable for p and Clairauts differential equation. Examples, problems based on these topics. (10 lectures)

UNIT-II

Differential equations of 2nd and 3rd order with constant coefficients of the type f(D)y = g(x), where g(x) e^{ax}, cosax, sinax, x^n their sum and products in pair. Problems based on these topics. (10 lectures)

UNIT-III

PDE of first order, linear equation of the form pP + qQ = R, Langrange's method. Non-linear first order equations, Charpit's method, PDE's of 2nd and 3rd order with constant co-efficients. Homogenous and Non-homogeneous partial differential equations. Examples and exercises based on these topics. (12 lectures)

UNIT-IV

General equation of sphere, Sphere through four points, Plane section of a sphere. Intersection of two spheres, Sphere with a given diameter, Intersection of a sphere and a line. Equation of tangent plane at any point of the sphere, Angle of intersection of two spheres, condition for the orthogonality of two spheres. (13 lectures)

UNIT-V

Equation of a cone with conic as guiding curve, enveloping cone of a sphere, condition that the given equation of 2nd degree should represents a cone, inter-section of a line with a cone, tangent plane to a cone at a point, condition for tangency for a plane, reciprocal cone, equation of a right circular cone, equation of cylinder, enveloping cylinder and equation of right circular cylinder. (15 lectures)

TEXTBOOKS:

1. Differential Calculus by Shanti Narayan, Dr. P X. Mittai, Pub, S. Chand.
2. S.L. Ross, Differential equations, Blaidell, Pub. Co.1994.
3. Solid Geometry by Shanti Narayan, Dr. P .K. Mittai. Pub. S. Chand.
4. I.N. Sneddon : Elements of partial differential equations, McGraw Book Company 1998
5. Jane Cronin : Differential Equations, Marcel Dekker, 1994
6. G.F. Simmons : Differential Equations with applications Tata McGraw Hill 1974.

Note for Paper Setter

1. Each lecture will be of one hour duration.
2. The question paper shall consist of 10 questions, two questions from each unit. The candidate will be required to do five questions selecting exactly one question from each unit.

Internal Assessment (Total Marks: 20)

20 marks for theory paper in a subject reserved for internal assessment shall be distributed as under:-

(i)	Class Test	:	10 marks
(ii)	Two Written Assignments/	:	10 marks

Unit-I	**Lesson No. 1**
B.A. 2nd Semester	**Subject : Mathematics**

DIFFERENTIAL EQUATIONS

Review

Differential Equations. As the name implies, a differential equation is an equation involving *differentials or derivatives.* Differential equations play a very dominant role in nearly every branch of science.

Types of differential Equations. There are two types of differential equations :

(*i*) Ordinary. (*ii*) Partial.

Definition. A differential equation involving a single independent variable and hence only ordinary derivatives, is called an ordinary differential equation. If there are two or more independent variables, so that the equation contains partial derivatives, it is called a partial differential equation.

The following are the examples of ordinary differential equations :

$$\frac{dx}{dt} = -kx, \quad k > 0 \qquad \text{...... (1)}$$

$$\frac{d^2 y}{dt^2} = g, \text{ (acceleration equation)} \qquad \text{...... (2)}$$

$$\frac{dv}{dt} = g, \text{ (velocity equation)} \qquad \text{...... (3)}$$

Typical examples of partial differential equations are Laplace's or the potential equation

$$\frac{\sigma^2 z}{\sigma x^2} + \frac{\sigma^2 z}{\sigma y^2} = 0, \ z = z\,(x, y) \qquad \text{...... (4)}$$

the diffusion or heat equation

$$\frac{\sigma^2 z}{\sigma x^2} = \frac{1}{k} \ \frac{\sigma z}{\sigma t}, \ z = z\,(x, \ t) \qquad \text{....... (5)}$$

and the wave equation :

$$\frac{\sigma^2 z}{\sigma x^2} = \frac{1}{c^2} \quad \frac{\sigma^2 z}{\sigma \ t^2}, \ z = z\,(x, \ t) \qquad \text{....... (6)}$$

Note : In equations (5) and (6), k and c^2 are certain constants. The equations (4) - (6), arise in a variety of problems in the fields of electricity and magnetism, elasticity and fluid mechanics.

Here we are primarily concerned with the study of ordinary differential equations.

Order and Degree. The *order* of an ordinary differential equation is the order of the highest derivative appearing in the equation, and the *degree* to be the exponent to which this highest derivative is raised when fractions and radicals involving y or its derivatives have been removed from the equation. (Students had already been introduced to these ideas in their Higher Secondary Syllabus) Equations (1) - (3) are of first degree, but the acceleration equation is of second order whereas the velocity equation and equation (1) are of first order.

Solution of a differential equation. A solution of a differential equation is a *relation* between the variables involved such that this relation and derivatives obtained there from satisfy the given differential equation.

Note : It can be easily verified that the first order equation (1) $\frac{dx}{dt} = -kx$ has solution $x(t)$ = c exp. ($-kt$), where c is an arbitracy constant.

Similarly, it can be shown that the function $y = \sin x$ and $y = \cos x$ are solutions of the differential equation.

$$\frac{d^2y}{dx^2} + y = 0 \text{ for all } x.$$

General Solution. The solution of a differential equation which involves as many arbitrary constants as the order of the differential equation, is called the general solution.

Particular Solution. The particular solution of a differential equation is that which is found from the general solution by giving. particular values to the arbitrary constants.

Example 1. Show that y=A cos x + B sin x is a solution of

$$\frac{d^2y}{dx^2} + y = 0$$

Solution. We have

$$y = A\cos x + B\sin x$$

$$\therefore \quad \frac{dy}{dx} = -A\sin x + B\cos x,$$

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x,$$

From above, we obtain $\frac{d^2y}{dx^2} + y = 0$, for all x.

Hence y = A cos x + B sin x is a solution of the given differential equation.

Note : We call y= A cos x + B sin x where A and B are arbitrary constants to be the *general solution* of $\frac{d^2y}{dx^2} + y = 0$. (The differential equation is being of order 2).

Formation of differential equations. Ordinary differential equations are associated with family of curves and are obtained by the elimination of the arbitrary constants, called parameters, from the given equation.

Example 1. Find the differential equation of the family of circles with centres on the x-axis.

Solution. The family of circles with centres on the x-axis is given by $(x-h)^2 + y^2 = a^2$, where a and h are the parameters.

Differentiating twice w.r.t. x, we obtain

$$2(x-h) + 2y\frac{dy}{dx} = 0 \qquad \text{... (1)}$$

and $$2 + 2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0 \qquad \text{... (2)}$$

The result (2) on simplification gives

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0$$

which is the required diffential equation.

Example 2. Find the differential equation of all ellipses centered at the origin.

Solution. The family of ellipses centered at the origin is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are the parameters.

Differentiating twice w.r.t. x, we obtain

$$\frac{2x}{a^2} + \frac{2y}{b^2} + \frac{dy}{dx} = 0 \qquad \text{... (1)}$$

and

$$\frac{2}{a^2} + \frac{2}{b^2}\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0 \qquad \text{... (2)}$$

From (1), $$\frac{b^2}{a^2} = -\frac{y}{x}\frac{dy}{dx} \qquad \text{... (3)}$$

From (2), $-\frac{y}{x}\frac{dy}{dx}+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$

or $xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$

which is the required differential equation.

Example 3. Find the differential equation of all parabolas whose axes are parallel to y-axis.

Solution. The family of parabolas whose axes are parallel to y-axis is

$x^2 + \text{D}x + \text{E}y + \text{F} = 0$

where D, E and F are the parameters.

Differentiating thrice w.r.t. x, we obtain

$2x + \text{D} + \text{E}\,\frac{dy}{dx}=0$,

$2 + \text{E}\,\frac{d^2y}{dx^2}=0$,

$\frac{d^3y}{dx^3}=0$

which is the required differential equation.

Example 4. From $x^2 + y^2 + 2gx + 2fy + c = 0$, derive a differential equation not containing g, f or c.

Solution. We have

$x^2 + y^2 + 2gx + 2fy + c = 0$

Differentiating thrice w.r.t. x, we obtain

$2x+2y\frac{dy}{dx}+2g+2f\,\frac{dy}{dx}=0$

or $(y+f)\frac{dy}{dx}=-(x+g)$

$$\text{or } \frac{dy}{dx} = -\frac{(x+g)}{(y+f)} \qquad \text{... (1)}$$

$$\frac{d^2y}{dx^2} = -\frac{(y+f)-(x+g)\dfrac{dy}{dx}}{(y+f)^2}$$

$$= -\frac{(y+f)+\dfrac{(x+g)^2}{y+f}}{(y+f)^2}$$

$$= -\frac{(y+f)^2+(x+g)^2}{(y+f)^3}$$

$$= -\frac{f^2+g^2-c}{(y+f)^3} \qquad \text{... (2)}$$

$$\frac{d^3y}{dx^3} = 3\frac{(f^2+g^2-c)}{(y+f)^4}\frac{dy}{dx} \qquad \text{... (3)}$$

From (1) and (3), we have

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3} = 3\left[1+\left(\frac{x+g}{y+f}\right)^2\right]\frac{f^2+g^2-c}{(y+f)^4}\frac{dy}{dx}$$

$$= 3\frac{(f^2+g^2-c)^2}{(y+f)^6}\frac{dy}{dx}$$

$$= 3\left(\frac{d^2y}{dx^2}\right)^2\frac{dy}{dx} \quad \text{[from (2)]}$$

$$\therefore \quad \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3} - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0$$

which is the required differential equation.

Example 5. Form a differential equation corresponding to $y^2 - 2ay + x^2 = a^2$ by eliminating a.

Solution. The given relation is

$$y^2 - 2ay + x^2 = a^2 \qquad \text{... (1)}$$

Differentiating w.r.t. x, we have

$$2y\frac{dy}{dx} - 2a\frac{dy}{dx} + 2x = 0$$

$$\text{or} \qquad y\frac{dy}{dx} - a\frac{dy}{dx} + x = 0$$

$$\text{or} \qquad a = \frac{y\frac{dy}{dx} + x}{\frac{dy}{dx}} \qquad \text{... (2)}$$

Substituting the value of a from (2) in (1), we get

$$y^2 - 2y\frac{\left(y\frac{dy}{dx} + x\right)}{\frac{dy}{dx}} + x^2 = \frac{\left(y\frac{dy}{dx} + x\right)^2}{\left(\frac{dy}{dx}\right)^2}$$

which on simplification gives

$$(x^2 - 2y^2)\left(\frac{dy}{dx}\right)^2 - 4xy\frac{dy}{dx} - x^2 = 0$$

which is the required differential equation.

Exercise 1

1. Determine the order and degree of each of the following equations.

(i) $\frac{dy}{dx} + y \cos x = \sin x$

(ii) $\frac{dy}{dx} + xy^2 = 0$

(iii) $\frac{d^2y}{dx^2} + y = 0$

(iv) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2y = \sin x$

(v) $x^2 \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + y = 0$

(vi) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 6xy = e^x$

(vii) $\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$

2. Verify that the following are solutions of the indicated differential equations :

(i) $y = C\,e^{-kx};\ \frac{dy}{dx} + kx = 0$

(ii) $y = 2x(e^x + C);\ x\frac{dy}{dx} - y = 2x^2e^x$

(iii) $y = C\,e^{\frac{y}{x}};\ \left(xy - x^2\right)\frac{dy}{dx} = y^2$

(iv) $y = \frac{A}{x} + B;\ \frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = 0$

(v) $y = A\ \cos x + B\ \sin x\ ;\ \frac{d^2y}{dx^2} + y = 0$

(vi) $y = A\ \cos x + B\ \sin x\ - \frac{x}{2}\ cos\ x;\quad \frac{d^2y}{dx^2} + y = \sin\ x$

3. Find the differential equation of the family of lines passing through the origin.

4. Find the differential equation of the family of circles with centres on the x-axis.

5. Find the differential equation of the family of parabolas with vertex at the origin and axis on the *x*-axis.

6. Find the differential equation of the family of hyperbolas having the coordinate axes as asymptotes.

7. Find the differential equation of the family of straight lines which are at a fixed distance *p* from the origin.

8. Find the differential equation :

(i) of all circles of radius *a*.

(ii) of all circles which pass through the origin and whose centers are on the *x*-axis.

(iii) that must be satisfied by the family of concentric circles $x^2—y^2=a^2$.

9. Find the differential equation of the system of circles touching the *y*-axis at the origin.

10. From the equation $y = ax^2 + bx + c,$ form a differential equation not containing *a, b or c.*

ANSWERS

1. (i) Order 1, degree 1. (ii) Order1, degree 1.

(iii) Order 2, degree 1. (iv) Order 2, degree 1.

(v) Order 3, degree 1. (vi) Order 3, degree 1.

(vii) Order 2, degree 2.

3. $x\frac{dy}{dx} = y$

4. $1 + y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$

5. $y = 2x\frac{dy}{dx}$ 6. $x\frac{dy}{dx} + y = 0$

7. $\left(x\frac{dy}{dx} - y\right)^2 = p\left[1 + \left(\frac{dy}{dx}\right)^2\right]$

8. (*i*) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

(ii) $x - y^2 + 2xy.\frac{dy}{dx} = 0$ (*iii*) $x + y\frac{dy}{dx} = 0$

9. $x^2 — y^2 + 2xy\ \frac{dy}{dx} = 0.$

10. $\frac{d^3y}{dx^3} = 0$

Initial and boundary-value problems.

Initial-value problem. Definition. An initial-value problemis a differential equation together with initial conditions. For example, the 'acceleration equation'.

$$\frac{d^2y}{dt^2} = g$$

together with initial conditions

$$y(0) = y_0, \frac{dy(0)}{dt} = u$$

comprise an 'initial-value' problem.

Boundary-value problem. Definition. A boundary-value problem is a differential equation together with boundary conditions. For example, the differential equation.

$$\frac{d^2x}{dt^2} = -\mu \cos\sqrt{\mu\ t}, \quad 0 \le t \le 2\pi$$

together with the boundary conditions

$$x(0) = 1, \quad x(2\pi) = -1$$

comprise a ‘boundary value’ problem.

These types of problems we come across in physics and engineering.

Art. Linear Equations

A differential equation of the form

$$\frac{dy}{dx} + Py = Q \qquad \text{... (1)}$$

where P and Q are constants or function of x alone is called a linear differential equation of the first order. It is so called because the dependent variable y and its derivative $\frac{dy}{dx}$ both appear in the equation in the first degree only.

Multiplying both sides of (1) by $e^{\int Pdx}$, we get

$$e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int Pdx} \qquad \text{... (2)}$$

$$\because \quad \frac{d}{dx}\left[ye^{\int Pdx}\right] = ye^{\int Pdx}.P + e^{\int Pdx}.\frac{dy}{dx}$$

$$= e^{\int Pdx}\left(\frac{dy}{dx} + Py\right)$$

We may write (2) as

$$\frac{d}{dx}\left[ye^{\int Pdx}\right] = Qe^{\int Pdx}$$

Integrating both sides w.r.t. x, we have

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C$$

as the complete solution of (1)

Note : The factor $e^{\int Pdx}$ on multiplying by which the left-hand member of (1) becomes the exact differential coefficient of some function of x and y is called an integrating factor (I.F.). It is useful to remember that after the intergrating factor has been introduced, the differential equation (1) takes the form.

$$\frac{d}{dx}(y \times I.F.) = Q \times I.F.$$

and can be directly integrated.

Cor. Sometimes a given differential equation can be reduced to linear form if we take x as the dependent variable instead of y. The standard form of the linear equation in this case is

$$\frac{dx}{dy} P_1 y = Q_1$$

where P_1 *and* Q_1 are constants or functions of y alone. The I.F. for this equation is $e^{\int P_1 dy}$

Note 2. In finding the I.F. no constant of integration need be added, as such a constant will not in any way make the solution more general than it is without such a constant.

Q.1. Solve the differential equation

$$\left(1+x^2\right)\frac{dy}{dx} + xy = \left(x+1\right)$$

Sol. The equation can be written as

$$\frac{dy}{dx} + \frac{x}{1+x^2} y = \frac{x+1}{1+x^2} \qquad \text{... (1)}$$

which is the linear equation in standard form.

Here $P = \frac{x}{1+x^2}$ $\qquad \therefore \qquad \int P dx = \int \frac{x}{1+x^2} dx$

$$= = \frac{1}{2} \int \frac{2x}{1+x^2} dx = \frac{1}{2} \log(1+x^2)$$

$$\therefore \quad I.F. = e^{\int P dx} = e^{\frac{1}{2} \log \ (1+x^2)} = e^{\log \ (1+x^2)^{\frac{1}{2}}}$$

$$= e^{\log \sqrt{1+x^2}} = \sqrt{1+x^2}$$

On multiplying by this factor, the equation (1) becomes

$$\sqrt{1+x^2} \left\{ \frac{dy}{dx} + \frac{x}{1+x^2} y \right\} = \frac{x+1}{\sqrt{1+x^2}}$$

$$or \quad \frac{d}{dx} \left\{ y\sqrt{1+x^2} \right\} = \frac{x+1}{\sqrt{1+x^2}}$$

on integrating w.r.t.x, we get

$$y\sqrt{1+x^2} = \int \frac{x+1}{\sqrt{1+x^2}} dx + C$$

$$= \frac{1}{2} \int \frac{2x}{\sqrt{1+x^2}} dx + \int \frac{1}{\sqrt{1+x^2}} dx + C$$

$$\therefore \quad y\sqrt{1+x^2} = \frac{1}{2} \frac{\left(1 = x^2\right)^{1/2}}{\frac{1}{2}} + \log \ \left\{ x + \sqrt{1+x^2} \right\} + C$$

$$or \quad y\sqrt{1+x^2} = \sqrt{1=x^2} + \log\left\{x+\sqrt{1=x^2}\right\} + C$$

which is the complete solution.

Q.2 Solve the differential equation :

$$\left(x+2y^3\right)\frac{dy}{dx} = y$$

Sol. The equation can be written as

$$y\frac{dy}{dx} = x + 2y^3$$

$$or \qquad \frac{dx}{dy} - \frac{1}{y}x = 2y^2$$

which is linear differential equation in x

Here $P = -\frac{1}{y}$ $\qquad \therefore \qquad \int Pdy = -\int\frac{dy}{y} = -\log\ y$

$$\therefore \qquad I.F. = e^{\int P\,dy} = e^{-\log\ y} = e^{\log\ y^{-1}} = y^{-1} = \frac{1}{y}$$

Hence multiplying by $\frac{1}{y}$, the equation becomes

$$\frac{1}{y}\frac{dx}{dy} - \frac{x}{y^2} = 2y$$

Integrating we have

$$\frac{x}{y} = \int = 2y\ dy\ +\ C$$

i.e. $\frac{x}{y} = y^2 + C$

i.e. $x = y^3 + cy$

which is the complete solution.

Q.3 Solve the differential equation.

$$\frac{dy}{dx} + y = e^{-x}$$

Sol. The =n is linear differential equation in y

$\therefore$ $I.F. = e^{\int dx} = e^x$

$\therefore$ the solution is

$$ye^x = \int e^x e^x dx + c$$

or $ye^x = \int dx + c$

or $ye^x = x + c$

Q.4 Solve the equation

$$\frac{dy}{dx} = (1-x) + (1-y)$$

Sol. The equation can be written as

$$\frac{dy}{dx} = 2 - x - y$$

i.e. $\frac{dy}{dx} + y = (2 - x)$

which is linear differential equation in y

i.e. $I.F. = e^{\int dx} = e^x$

$\therefore$ the solution is

$$ye^x = \int (2 - x) e^x dx + c$$

$$= 2e^x - \left[xe^x - \int e^x dx \right] + c$$

Integrating by parts

$$= 2e^x - xe^x + e^x + c$$

$$= (3 - x)\ e^x + c$$

or $(x + y - 3)\ e^x = c$

or $x + y - 3 = c\ e^{-x}$

Q.5 Solve the differential equation

$$\left(1 - x^2\right)\frac{dy}{dx} + 2xy = x\sqrt{1 - x^2}$$

Sol. The equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1 - x^2}\, y = \frac{x\sqrt{1 - x^2}}{1 - x^2}$$

i.e $\dfrac{dy}{dx} + \dfrac{2x}{1 - x^2}\, y = \dfrac{x}{\sqrt{1 - x^2}}$

Here $P = \dfrac{2x}{1 - x^2}$

$\therefore$ $\int P dx = -\int \dfrac{-2x}{1 - x^2} dx = -\log\left(1 - x^2\right)$

$$= \log\left(1 - x^2\right)^{-1}$$

$$\therefore \quad I.F. = e^{\int P dx} = e^{\log\left(1-x^2\right)^{-1}} = \left(1 - x^2\right)^{-1}$$

$$= \frac{1}{1 - x^2}$$

$\therefore$ the solution is

$$y \times \frac{1}{1 - x^2} = \int \frac{x}{\sqrt{1 - x^2}} \times \frac{1}{1 - x^2}\, dx + c$$

$$= \int \frac{x}{(1 - x^2)^{3/2}}\, dx + c$$

$$= -\frac{1}{2} \int \frac{-2x}{(1 - x^2)^{3/2}}\, dx + c$$

$$= -\frac{1}{2} \frac{\left(1 - x^2\right)^{-\frac{1}{2}}}{-\frac{1}{2}} + c$$

$$= \left(1 - x^2\right)^{-\frac{1}{2}} + c$$

$$= \frac{1}{\sqrt{1 - x^2}} + c$$

$$\therefore \quad \frac{y}{1 - x^2} = \frac{1}{\sqrt{1 - x^2}} + c$$

or $\quad y = \sqrt{1 - x^2} + c\,(1 - x^2)$

Q.6 Solve the differential equation

$$\frac{dy}{dx} + y \cos x = \sin x \cos x$$

Sol. The differential equation is linear in y

$\therefore \quad I.F. = e^{\int \cos x\, dx} = e^{\sin x}$

$\therefore$ the solution is

$$y e^{\sin x} = \int \sin x \cos x\, e^{\sin x} dx + c$$

Put $\quad \sin x = z$

$\therefore \quad \cos x\, dx = dz$

$\therefore$ the solution is

$$y\, e^{z} = \int z e^{z} dz + c$$

Integrating by parts, we have

$$y\, e^{z} = z\, e^{z} - \int e^{z} dz + c$$

$\therefore \quad y\, e^{z} = z\, e^{z} - e^{z} + c$

$\therefore \quad y = z - 1 + c\, e^{-z}$ where $z = \sin x$

Q.7 Solve the differential equation

$$(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$$

Sol. The equation can be written as

$$\frac{dx}{dy} = \frac{\tan^{-1} y}{1 + y^2} - \frac{x}{1 + y^2}$$

or $\dfrac{dx}{dy}+\dfrac{x}{1+y^2}=\dfrac{\tan^{-1}y}{1+y^2}$

which is linear differential equation in *x*.

Here $P=\dfrac{1}{1+y^2}$

$\therefore$ I.F. = $e^{\int P dy}=e^{\int \frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$

$\therefore$ the solution is

$$xe^{\tan^{-1}y}=\int \frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}dy+c$$

Put $\tan^{-1}y=z$

$\therefore$ $\dfrac{1}{1+y^2}dy=dz$

$\therefore$ the solution is

$$x\,e^z=\int z\,e^z dz+c$$

$=ze^z-\int e^z dz+c$, Integrating by parts

$=ze^z-e^z+c$

or $x=z-1+c\,e^{-z}$ *where* $z=\tan^{-1}y$

Q.8 Solve the differential equation

$$1+\left(x\tan y-\sec y\right)\frac{dy}{dx}=0$$

Sol. The $=n$ can be written as

$$\frac{dx}{dy}+x\tan y-\sec y=0$$

or $\dfrac{dx}{dy} + x\tan y = \sec y$

which is linear differential =n in x

$$\therefore \qquad I.F. = e^{\int \tan y\, dy} = e^{\int \frac{\sin y}{\cos y} dy}$$

$$= e^{-\int \frac{-\sin y}{\cos y} dy} = e^{-\log \cos y} = e^{\log \sec y}$$

$= \sec y.$

$\therefore$ The solution is

$$x \sec y = \int \sec y\, sec\, y\, dy + c$$

or $x \sec y = \int \sec^2 y dy + c$

or $x \sec y = \tan y + c$

or $\dfrac{x}{\cos y} = \dfrac{\sin y}{\cos y} + c$

or $x = \sin y + \text{c} \cos y$

Q.9 Solve the differential equation

$$\left(1 - x^2\right)\frac{dy}{dx} = x\left(\sin^{-1} x - y\right)$$

Sol. The equation can be written as

$$\frac{dy}{dx} = \frac{x\sin^{-1}x}{1 - x^2} - \frac{xy}{1 - x^2}$$

or $\dfrac{dy}{dx} + \dfrac{x}{1 - x^2}y = \dfrac{x\sin^{-1}x}{1 - x^2}$

which is a linear differential equation in y

$$\therefore \quad I.F. = e^{\int \frac{x}{1-x^2}dx} = e^{-\frac{1}{2}\int \frac{-2x}{1-x^2}dx}$$

$$= e^{-\frac{1}{2}\log\left(1-x^2\right)} = e^{\log\left(1-x^2\right)^{-\frac{1}{2}}}$$

$$\therefore \quad I.F. = \left(1-x^2\right)^{-1/2} = \frac{1}{\sqrt{1-x^2}}$$

$\therefore$ the solution is

$$\frac{y}{\sqrt{1-x^2}} = \int \frac{x\ \sin^{-1} x}{1-x^2}\frac{1}{\sqrt{1-x^2}}dx + c$$

or $$\frac{y}{\sqrt{1-x^2}} = \int \frac{x\ \sin^{-1} x}{\left(1-x^2\right)^{3/2}}dx + c$$

In R.H.S. Put $x = \sin\ \theta$

$$\because \quad dx = \cos\theta\ d\theta$$

$$\therefore \frac{y}{\sqrt{1-x^2}} = \int \frac{\sin\theta(\sin^1 \sin^- \theta)}{\left(1-\sin^2\theta\right)^{3/2}}\cos\theta\, d\theta + c$$

$$= \int \frac{\theta\ \ \sin\ \theta}{\cos^3\ \theta}\cos\theta\ \ d\theta + c$$

$$= \int \frac{\theta\ \ \sin\ \theta}{\cos\ \theta}\frac{1}{\cos\ \theta}\ d\theta + c$$

$$= \int \theta\ \tan\ \theta\ \sec\ \theta\ d\theta + c$$

$$= \theta.\ sec\ \theta - \int \sec\ \theta\ d\theta + c,\ \text{Integrating by parts}$$

$$= \theta\ sec\ \theta - \int \frac{\sec\ \theta\ (\sec\theta + \tan\theta)}{\sec\theta + \tan\theta} d\theta + c$$

$$= \theta\ sec\ \theta - \int \frac{\sec\ \theta\ tan\ \theta\ + \sec^2\theta}{\sec\theta + \tan\theta} d\theta + c$$

$$= \theta\ sec\ \theta - \log\ (\sec\theta + \tan\theta) + c$$

Now $x = \sin\ \theta$

$\therefore \quad cos\ \theta = \sqrt{1 - x^2}$

$\therefore \quad sec\ \theta = \frac{1}{\sqrt{1 - x^2}}\ and\ tan\ \theta = \frac{x}{\sqrt{1 - x^2}}$

$\therefore$ The solution is

$$\frac{y}{\sqrt{1-x^2}} = sin^{-1}x.\frac{1}{\sqrt{1-x^2}} - \log\left[\frac{1}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}}\right] + c$$

$$= \frac{sin^{-1}x}{\sqrt{1-x^2}} - \log\left(\frac{1+x}{\sqrt{1-x^2}}\right) + c$$

$$= \frac{sin^{-1}x}{\sqrt{1-x^2}} - \log\frac{1+x}{\sqrt{(1-x)(1+x)}} + c$$

$$= \frac{sin^{-1}x}{\sqrt{1-x^2}} - \log\left(\frac{\sqrt{(1-x)}}{(1-x)}\right) + c$$

$$= \frac{sin^{-1}x}{\sqrt{1-x^2}} + \frac{1}{2}\ \log\frac{1-x}{1+x} + c$$

$$\therefore \quad y = sin^{-1}x + \left[c + \frac{1}{2}\log\frac{1-x}{1+x}\right]\sqrt{1-x^2}$$

Q.10 Solve the differential equation

$$\cos^2 y\frac{dy}{dx} + y = \tan x$$

Sol. The equation can be written as

$$\frac{dy}{dx} + \frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$

or $$\frac{dy}{dx} + y\ \sec^2 x = \tan\ x\ sec^2 x$$

which is linear equation in y

$$\therefore \quad I.F. = e^{\int \sec^2 x\, dx} = e^{\tan\ x}$$

$\therefore$ The solution is

$$y\, e^{\tan\ x} = \int e^{\tan x}.\tan x\ \sec^2 x\, dx + c$$

Put $\tan x = t$

$$\therefore \quad sec^2 x\, dx = dt$$

$\therefore$ The solution is

$$y\, e^t = \int te^t dt + c$$

$= te^t - \int e^t dt + c$, Integrating by parts.

$$\therefore \quad y\, e^t = te^t - e^t + c$$

$$\therefore \quad y = t - 1 + ce^{-t}$$

$\Rightarrow \qquad y = \tan x - 1 + ce^{-\tan x}$

Q.11 Solve the differential equation

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1$$

Sol. The equation can be written as

$$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}$$

$$\Rightarrow \qquad \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

which is linear differential equation in *y*.

$\therefore$ I.F. $= e^{\int \frac{1}{\sqrt{x}}dx} = e^{\int x^{-1/2}dx} = e^{\frac{x^{1/2}}{\frac{1}{2}}} = e^{2\sqrt{x}}$

$\therefore$ the solution is

$$yx^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} e^{2\sqrt{x}} dx + c$$

i.e. $\quad yx^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}} dx + c$

$$= \int x^{-1/2} dx + c$$

$$= \frac{\frac{x^{1/2}}{1} + c}{2}$$

$\therefore \quad ye^{2\sqrt{x}} = 2\sqrt{x} + c$

Q.12 Solve the differential equation.

$$\left(1+y+x^2y\right)\frac{dx}{dy}+\left(x+x^3\right)=0$$

Sol. The equation can be written as

$$\left(x+x^3\right)\frac{dy}{dx}+1+y+x^2y=0$$

or $\quad x\left(1+x^2\right)\frac{dy}{dx}+(1+x^2)y=-1$

or $\quad \frac{dy}{dx}+\frac{1}{x}y=-\frac{1}{x(1+x^2)}$

which is linear differential equation in y

$$\therefore \quad I.F. = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

$\therefore$ the solution is

$$yx = \int -\frac{1}{x(1+x^2)}x\,dx + c$$

$$= -\int \frac{1}{\left(1+x^2\right)}dx + c$$

$$= - \tan^{-1}x + c$$

$$\therefore \quad y = \frac{-\tan^{-1}x + c}{x}$$

Q.13 Solve the differential equation

$$\frac{dy}{dx}+\frac{y}{(1-x^2)^{3/2}}=\frac{x+\sqrt{1-x^2}}{(1-x^2)^2}$$

Sol. The equation is linear in y

$\therefore \quad I.F.=e^{\int \frac{1}{(1-x^2)^{3/2}}dx}$

Put $x=\sin\theta$

$\therefore \quad dx=\cos\theta\, d\theta$

$\therefore \quad I.F.=e^{\int \frac{1}{(1-\sin^2\theta)^{3/2}} \cos\theta\, d\theta}$

$=e^{\int \frac{1}{\cos^3\theta} \cos\theta\, d\theta}$

$=e^{\int \frac{1}{\cos^2\theta} d\theta}=e^{\int sec^2\theta\, d\theta}$

$=e^{\tan\theta}$

But $x=\sin\theta$

$\therefore \quad \cos\theta=\sqrt{1-x^2}$

$\therefore \quad \tan\theta=\frac{x}{\sqrt{1-x^2}}$

$\therefore \quad I.F.=e^{\frac{x}{\sqrt{1-x^2}}}$

$\therefore$ the solution is

$$\therefore \quad ye^{\frac{x}{\sqrt{1-x^2}}}=\int\frac{x+\sqrt{1-x^2}}{(1-x^2)^2}e^{\frac{x}{\sqrt{1-x^2}}}dx+c$$

$$= \int \frac{\sin\ \theta + cos\ \theta}{(1-\sin^2\theta)^2} e^{\tan\ \theta} \cos\ \theta\ d\theta\ + c$$

$$= \int \frac{\sin\ \theta + cos\ \theta}{\cos^4\theta} e^{\tan\ \theta} \cos\ \theta\ d\theta\ + c$$

$$\therefore \qquad ye^{\tan\ \theta} = \int \left(\frac{\sin\ \theta}{\cos\ \theta} + 1\right) e^{\tan\ \theta}\ sec^2\theta\ d\theta\ + c$$

$$= \int (tan\ \theta + 1)\, e^{\tan\ \theta}\ sec^2\theta\ d\theta\ + c$$

Put $\tan\theta = z$

$\therefore\ \sec^2\theta\ d\theta = dz$

$$\therefore \qquad ye^z = \int (z+1)\, e^z\ dz\ + c$$

$$= \int ze^z dz + \int e^z dz\ + c$$

$$= ze^z - \int e^z dz\ + \int e^z dz + \ c$$

$$\therefore \qquad ye^z = ze^z - c$$

$$\therefore \qquad y = z - ce^{-z}$$

$$= \tan\theta - ce^{-\tan\theta}$$

$$\therefore \qquad y = \frac{x}{\sqrt{1-x^2}} - ce^{\frac{-x}{\sqrt{1-x^2}}}$$

Q.14 Integrate $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ and obtain the equation of the cubic curve satisfying this equation and passing through the origin.

Sol. The equation can be written in the form

$$(1+x^2)\frac{dy}{dx}+2xy=4x^2$$

or $$\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{4x^2}{1+x^2}$$

which is linear differential equation in y

Here $$P=\frac{2x}{1+x^2}$$

$$\therefore \int Pdx=\int\frac{2x}{1+x^2}dx=\log\ (1+x^2)$$

$$\therefore \quad I.F.=e^{\int Pdx}=e^{\log\ (1+x^2)}=1+x^2$$

$\therefore$ the solution is

$$y(1+x^2)=\int\frac{4x^2}{(1+x^2)}\times(1\times x^2)\ dx+c$$

or $$y(1+x^2)=\int 4x^2\ dx+c$$

or $$y(1+x^2)=\frac{4x^3}{3}+c$$

If this curve passes through (0, 0)

$\therefore$ 0 = 0 + c

$\Rightarrow$ c = 0

$\therefore$ the solution is

$$y(1+x^2)=\frac{4x^3}{3}$$

or $3y(1 + x^2) = 4x^3$

Q.15 Solve the differential equation

$$(2x + 10y^3)\,\frac{dy}{dx} + y = 0$$

Sol. The equation may be written as

$$y\frac{dx}{dy} + 2x - 10y^3 = 0$$

or $$\frac{dx}{dy} + \frac{2}{y}x = 10\,y^2$$

The differential equation is linear in x

Here $\quad P = \dfrac{2}{y}$

$\therefore \quad \int P\,dy = \int \frac{2}{y}\,dy = 2\log y = \log y^2$

$\therefore \quad I.F. = e^{\int P\,dy} = e^{\log y^2} = y^2$

$\therefore$ the solution is

$$xy^2 = \int 10y^2 . y^2\,dy + c$$

or $$xy^2 = \int 10y^4\,dy + c$$

$$= \frac{10y^5}{5} + c$$

$\Rightarrow \; xy^2 = 2y^5 + c$

Q.16 Solve the differential equation

$$\frac{dy}{dx}+\frac{1}{x}y=e^{x}$$

Sol. The differential equation is linear in y

$\therefore \quad I.F.=e^{\int \frac{1}{x}dx}=e^{\log x}=x$

$\therefore$ the solution is

$$yx=\int xe^{x}dx+c$$

$=xe^{x}-\int e^{x}dx+c$, Integrating by parts.

$\therefore \quad yx=xe^{x}-e^{x}+c$

Q.17 Solve the differential equation

$$x\,log\,x\,\frac{dy}{dx}+y=\;2\,log\,x$$

Sol. This can be written in the form

$$\frac{dy}{dx}+\frac{y}{x\,log\,x}=\;\frac{2}{x}$$

The differential equation is linear in y

$\therefore \quad I.F.=e^{\int \frac{1}{x\,log\,x}dx}=e^{\log x}=\log\;x$

$\therefore$ the solution is

$$y\log x=\int \frac{2}{x}\log x\,dx+c$$

Put log $x = t$ $\therefore$ $\frac{1}{x} dx = dt$

$\therefore$ $y \log x = \int 2t\, dt + c$

$= t^2 + c$

$\therefore$ $y \log x = (\log x)^2 + c$

Q.18 Solve the differential equation

$$(1 + x^2)\frac{dy}{dx} + 2xy = \cos x$$

Sol. The =*n* can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2} \quad y = \frac{\cos x}{1+x^2}$$

which is linear differential equation in *y*

$\therefore$ $I.F. = e^{\int \frac{2x}{1+x^2} dx} = e^{\log (1+x^2)} = 1 + x^2$

$\therefore$ the solution is

$$y(1 + x^2) = \int \frac{\cos x}{1+x^2} \times (1+x^2)\, dx + c$$

$$= \int \cos x\, dx + c$$

$\therefore$ $y(1 + x^2) = \sin x + c \Rightarrow y(1 + x^2) = \sin x + c$

\- - - - - -

Art. Bernouilli's Equation :-

The equation

$$\frac{dy}{dx} + Py = Q\,y^n \qquad \text{... (1)}$$

where P and Q are constants or functions of *x* alone and *n* is a constant other than zero or unity, is called Bernouilli's equation or the extended linear equation. This equation can be easily transformed into the linear form by means of the substitution $z = y^{1-n}$ and then solved by the method of the previous section (Lesson) i.e. linear form.

Dividing throughout by y^n, (1) becomes

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \qquad \text{... (2)}$$

Put $y^{1-n} = z$

$$\therefore \quad (1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ equation (2) is transformed to

$$\frac{dz}{dx} + (1-n)\,P_z = (1+n)\,Q$$

which is linear in *z* and can be easily solved.

The I.F. in this case is e $^{(1-n)Pdx}$

Q.1 Solve the differential equation

$x\,dy = y(1 + xy)\,dx$

Sol. The equation may be written as

$$x\frac{dy}{dx} = y + xy^2$$

or $$y^{-2}\frac{dy}{dx} - \frac{y^{-1}}{x} = 1$$

or $$-y^{-2}\frac{dy}{dx} + \frac{y^{-1}}{x} = -1$$

Put $y^{-1} = z$

$$\therefore \quad -y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ the equation is transformed to

$$\frac{dz}{dx} + \frac{z}{x} = -1$$

which is linear differential equation in *z*

$$\therefore \quad I.F. = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$\therefore$ the solution is

$$xz = \int -x\,dx + c$$

$$= -\frac{x^2}{2} + c$$

or $xy^{-1} = -\frac{x^2}{2} + c$

or $\frac{x}{y} = -\frac{1}{2}x^2 + c$

or $x = -\frac{1}{2}x^2 y + cy$

Q.2 Solve the differential equation

$(1 + x^2)\left(\frac{dy}{dx}\right) — 4x^2 \cos^2 y + x \sin 2y = 0$

Sol. The equation may be written as

$$\frac{dy}{dx} + \frac{x \sin 2y}{1+x^2} = \frac{4x^2}{1+x^2}\cos^2 y$$

or $\frac{dy}{dx} + \frac{2x}{1+x^2}\sin y \cos y = \frac{4x^2}{1+x^2}\cos^2 y$

or $\sec^2 y \frac{dy}{dx} + \frac{2x}{1+x^2}\tan y = \frac{4x^2}{1+x^2}$

Put $\tan y = z$

$\therefore$ $\sec^2 y \frac{dy}{dx} = \frac{dz}{dx}$

Then the equation transforms into

$$\frac{dz}{dx} + \frac{2x}{1+x^2} z = \frac{4x^2}{1+x^2}$$

which is linear differential equation in z

$$\therefore \quad I.F. = e^{\int \frac{2x}{1+x^2}dx} = e^{\log\,(1+x^2)} = 1+x^2$$

$\therefore$ the equation is

$$z(1+x^2) = \int \frac{4x^2}{1+x^2}(1+x^2)dx + c$$

$$= \int 4x^2 dx + c$$

or $(1 + x^2) \tan y = \dfrac{4x^3}{3} + c$

Q.3 Solve the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = y^2$$

Sol. The equation can be written as

$$y^{-2}\frac{dy}{dx} + \frac{y^{-1}}{x} = 1$$

Put $y^{-1} = z$

$$\therefore \quad -y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\therefore \quad y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}$$

$\therefore$ the equation is transformed to

$$-\frac{dz}{dx}+\frac{z}{x}=1$$

or $$\frac{dz}{dx}-\frac{z}{x}=-1$$

which is linear differential equation in z.

$\therefore$ I.F $= e^{\int -\frac{3}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$

$\therefore$ the solution is

$$\frac{z}{x}=\int -\frac{1}{x}dx+\log c$$

or $$\frac{y^{-1}}{x} = — \log x + \log c$$

or $$\frac{1}{xy}=\log\frac{c}{x}$$

$\therefore$ $1 = \text{xy} \log \frac{c}{x}$

or $xy \log \frac{c}{x} = 1$

Q.4 Solve the differential equation

$$\frac{dy}{dx}=x^3y^3 — xy$$

Sol. The =*n* can be written as

$$\frac{dy}{dx}+xy=x^3y^3$$

or $y^{-3}\dfrac{dy}{dx} + xy^{-2} = x^3$

Put $y^{-2} = z$

$\therefore \quad -2y^{-3}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

or $\quad y^{-3}\dfrac{dy}{dx} = -\dfrac{1}{2}\dfrac{dz}{dx}$

$\therefore$ the equation is transformed to

$$-\frac{1}{2}\frac{dz}{dx} + xz = x^3$$

or $\quad \dfrac{dz}{dx} - 2xz = -2x^3$

which is linear differential equation in z

$\therefore \quad I.F. = e^{\int -2x\,dx} = e^{-x^2}$

$\therefore$ the solution is

$$ze^{-x^2} = \int -2x^3 e^{-x^3}\,dx + c$$

Put $-x^2 = t$

$\therefore \quad -2x\,dx = dt$

and $\quad x^2 = -t$

$\therefore$ the solution is

$$z\,e^t = \int -t\,e^t\,dt + c$$

$$= -te^t + \int e^t\,dt + c$$

$= -te^t + e^t + c$

$\therefore \quad y^{-2} = -t + 1 + c\,e^{-t}$

or $\quad y^{-2} = 1 + x^2 + c\,e^{x^2}$

Q.5 Solve the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x^3 y^4$$

Sol. The equation can be written as

$$y^{-4}\frac{dy}{dx} + \frac{y^{-3}}{x} = x^3$$

Put $y^{-3} = z$

$$\therefore \quad -3y^{-4}\frac{dy}{dx} = \frac{dz}{dx}$$

or $$y^{-4}\frac{dy}{dx} = -\frac{1}{3}\frac{dz}{dx}$$

$\therefore$ the equation is transformed to

$$-\frac{1}{3}\frac{dz}{dx} + \frac{z}{x} = x^3$$

or $$\frac{dz}{dx} - \frac{3}{x}z = -3x^3$$

which is linear differential equation in z

$$\therefore \quad I.F. = e^{\int -\frac{3}{x}dx} = e^{-3\log x} = e^{\log x^{-3}} = x^{-3} = \frac{1}{x^3}$$

$\therefore$ the solution is

$$\frac{z}{x^3 =} \int -3x^3 \times \frac{1}{x^3} dx + c$$

or $\dfrac{y^{-3}}{x^3} = \int -3 \; dx + c$

or $\dfrac{1}{x^3 y^3} = -3x + c$

or $1 = x^3 y^3 \, (c—3x)$

$\Rightarrow$ $x^3 y^3 \, (c—3x) = 1$

Q.6 Solve the differential equation

$$\frac{dy}{dx} + \frac{x}{1+x^2} y = x\sqrt{y}$$

Sol. The equation can be written as

or $y^{-\frac{1}{2}} \dfrac{dy}{dx} + \dfrac{x}{1-x^2} \; y^{\frac{1}{2}} = x$

Put $y^{1/2} = z$

$\therefore$ $\dfrac{1}{2} y^{-1/2} \dfrac{dy}{dx} = \dfrac{dz}{dx}$

or $y^{-1/2} \dfrac{dy}{dx} = 2 \dfrac{dz}{dx}$

$\therefore$ the equation is transformed to

$$2\frac{dz}{dx}+\frac{x}{1-x^2}z=x$$

or $$\frac{dz}{dx}+\frac{1}{2}\frac{x}{1-x^2}z=\frac{1}{2}x$$

which is linear differential equation in z

$\therefore \quad I.F.=e^{\int\frac{1}{2}\frac{x}{1-x^2}dx}$

$$=e^{\int\frac{-1}{4}\frac{-2x}{1-x^2}dx}$$

$$=e^{-\frac{1}{4}\log(1-x^2)}=e^{\log\left(1-x^2\right)^{-\frac{1}{4}}}=(1-x^2)^{-\frac{1}{4}}$$

$\therefore$ the solution is

$$z(1-x^2)^{-\frac{1}{4}}=\int\frac{1}{2}x(1-x^2)^{-\frac{1}{4}}dx+\frac{c}{3}$$

$$=-\frac{1}{4}\int-2x(1-x^2)^{-\frac{1}{4}}dx+\frac{c}{3}$$

$$=-\frac{1}{4}\frac{(1-x^2)^{3/4}}{\frac{3}{4}}+\frac{c}{3}$$

or $$\sqrt{y}\,(1-x^2)^{-\frac{1}{4}}=-\frac{1}{3}(1-x^2)+\frac{c}{3}$$

or $$\sqrt{y}=-\frac{1}{3}(1-x^2)+\frac{c}{3}(1-x^2)^{\frac{1}{4}}$$

or $$3\sqrt{y}=-(1-x^2)+c(1-x^2)^{\frac{1}{4}}$$

or $1 - x^2 + 3\sqrt{y} = c(1-x^2)^{\frac{1}{4}}$

Q.7 Solve the differential equation

$$\frac{dy}{dx}(x^2y^3 + xy) = 1$$

Sol. The equation can be written as

$$\frac{dx}{dy} = x^2y^3 + xy$$

or $$\frac{dx}{dy} - xy = x^2y^3$$

or $$x^{-2}\frac{dx}{dy} - x^{-1}y = y^3$$

Put $x^{-1} = z$

$$\therefore \quad -x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$$

$$\Rightarrow \quad x^{-2}\frac{dx}{dy} = -\frac{dz}{dy}$$

$\therefore$ the equation is transformed to

$$-\frac{dz}{dy} - yz = y^3$$

$$\Rightarrow \quad \frac{dz}{dy} + yz = -y^3$$

which is linear equation in z

$$\therefore \quad I.F. = e^{\int y\,dy} = e^{\frac{y^2}{2}}$$

$\therefore$ the solution is

$$z\, e^{\frac{y^2}{2}} = \int -y^3 e^{\frac{y^2}{2}}\, dy + c$$

$$= -\int y . y^2 e^{\frac{y^2}{2}}\, dy + c$$

Put $\frac{y^2}{2} = t$

$\therefore$ $y^2 = 2\, t$

$\therefore$ $\frac{1}{2} .2\, y\, dy = dt$

or $y\, dy = dt$

$\therefore$ $z\, e^t = —\int\ 2t\, .\, e^t\, dt + c$

$= — 2 \int\ t\, e^t\, dt + c$

$\therefore$ $x^{—1}\, e^t = —2[\, t\, e^t — e^t] + c$, Integrating by part.

or $e^t = — 2x \left[\frac{y^2}{2} e^t - e^t \right] + cx$

$= x\, (—y^2 + 2)\, e^t + c\, x$

$\Rightarrow$ $x\, (2 — y^2)\, e^t + c\, x = e^t$ where $t = \frac{y^2}{2}$

Q.8 Solve the differential equation

$$\frac{dy}{dx} = 2y\ \tan x + y^2\ \tan^2 x$$

Sol. The equation can be written as

$$\frac{dy}{dx} - 2y\ \tan x = y^2\ \tan^2 x$$

or $$y^{-2}\frac{dy}{dx} - 2y^{-1}\ \tan x = \tan^2 x$$

Put $-y^{-1} = z$

$\therefore$ $$y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ the equation is transformed to

$$\frac{dz}{dx} + 2z\ tan\ x\ =\ tan^2 x$$

which is linear differential equation in z

$\therefore$ $$I.F. = e^{\int 2\ \tan x\ dx} = e^{-2\int \frac{-\sin x}{\cos x}dx}$$

$$= e^{-2\ \log\ \cos x}$$

$$= e^{2\ \log\ sec\ x}$$

$$= e^{\log\ sec^2 x} = \sec^2 x$$

$\therefore$ the solution is

$$z \sec^2 x = \int\ \tan^2 x \sec^2 x\ dx + c'$$

$\therefore$ $$-y^{-1} \sec^2 x = \frac{\tan^3 x}{3} + c'$$

or $\frac{-\sec^2 x}{y} = \frac{\tan^3 x}{3} - C$, where $c' = — c$

$\therefore \quad \sec^2 x = y(c — \frac{1}{3}\tan^3 x)$

$\sec^2 x = y(c — \frac{1}{3}\tan^3 x)$, say $c' = — c,$

$\Rightarrow \quad y(c — \frac{1}{3}\tan^3 x) = \sec^2 x$

Q.9 Solve the differential equation

$$3\frac{dy}{dx} + \frac{2y}{x+1} = \frac{x^3}{y^2}$$

Sol. The equation can be written as

$$3y^2\frac{dy}{dx} + \frac{2y^3}{x+1} = x^3$$

Put $y^3 = z$

$$\therefore \quad 3y^2\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ the equation is transformed to

$$\frac{dz}{dx} + \frac{2z}{x+1} = x^3$$

which is linear differential equation in z

$$\therefore \quad I.F. = e^{\int \frac{2}{x+1}dx} = e^{2\log(x+1)}$$

$$= e^{\log(x+1)^2} = (x+1)^2$$

$\therefore$ the solution is

$$z(x+1)^2 = \int x^3 (x+1)^2 \, dx + c$$

$$\therefore \quad y^3(x+1)^2 = \int x^3(x^2+2x+1)dx + c$$

$$\Rightarrow \quad y^3(x+1)^2 = \int (x^5+2x^4+x^3)\, dx + c$$

$$= \frac{x^6}{6} + \frac{2x^5}{5} + \frac{x^4}{4} + c$$

$$\Rightarrow \quad y^3(x+1)^2 = \frac{1}{6}x^6 + \frac{2}{5}x^5 + \frac{1}{4}x^4 + c$$

Q.10 Solve the differential equation

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$$

Sol. The equation can be written as

$$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$$,Multiplying throughout by $\cos y$

Put $+\sin y = z$

$$\therefore \quad \cos y \frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ the equation is transformed to

$$\frac{dz}{dx} - \frac{z}{1+x} = (1+x)e^x$$

which is different equation linear in z

$$\therefore \quad \text{I.F.} = e^{\int -\frac{1}{1+x} dx} = e^{-\log\,(1+x)}$$

$=e^{\log(1+x)-1}=(1+x)^{-1}=\dfrac{1}{1+x}$

$\therefore$ the solution is

$$\frac{z}{1+x}=\int(1+x)e^x\cdot\frac{1}{1+x}\,dx+c$$

$$=\int e^x dx+c$$

$\therefore$ $\dfrac{\sin y}{1+x}=e^x+c$

$\therefore$ $\sin y=(1+x)(e^x+c)$

Q. 11. Solve the differntial equation

$ydy+by^2dx=a\cos x\,dx$

Sol. The equation can be written as

$$y\frac{dy}{dx}+by^2=a\cos x$$

Put $y^2=z$

$\therefore$ $2y\dfrac{dy}{dx}=\dfrac{dz}{dx}$

$\Rightarrow$ $y\dfrac{dy}{dx}=\dfrac{1}{2}\dfrac{dz}{dx}$

$\therefore$ the equation is transformed to

$$\frac{1}{2}\frac{dz}{dx}+bz=a\cos x$$

or $\dfrac{dz}{dx}+2bz=2a\cos x$

which is linear differential equation in z

$\therefore$ I.F.$= e^{\int 2bdx} = e^{2bx}$

$\therefore$ the solution is

$ze^{2bx} = \int 2ae^{2bx} \cos x \, dx + c$

$\therefore$ $y^2 e^{2bx} = 2a \int e^{2bx} \cos x \, dx + c$

Now we have to integrate $\int e^{2bx} \cos x \, dx$

Integrating by parts, we have

$$\int e^{2bx} \cos x \, dx = \frac{e^{2bx}}{2a} \cos x - \int \frac{e^{2bx}}{2b} (-\sin x) \, dx$$

$$= \frac{e^{2bx} \cos x}{2b} + \frac{1}{2b} \int e^{2bx} \sin x dx$$

$$= \frac{e^{2bx} \cos x}{2b} + \frac{1}{2b} \left[\frac{e^{2bx}}{2b} \sin x - \int \frac{e^{2bx}}{2b} \cos x dx \right]$$

$$= \frac{e^{2bx} \cos x}{2b} + \frac{e^{2bx}}{4b^2} \sin x - \frac{1}{4b^2} \int e^{2bx} \cos x dx$$

$\therefore$ $$\left(1 + \frac{1}{4b^2}\right) \int e^{2bx} \cos x \, dx = \frac{e^{2bx}(2b \cos x + \sin x)}{4b^2}$$

or $$\left(\frac{4b^2 + 1}{4b^2}\right) \int e^{2bx} \cos x \, dx = \frac{e^{2bx}(2b \cos x + \sin x)}{4b^2 + 1}$$

$$\therefore \quad \int e^{2bx} \cos x \, dx = \frac{e^{2bx}(2b\cos x + \sin x)}{4b^2+1}$$

$\therefore$ the solution is

$$y^2 e^{2bx} = \frac{2a\, e^{2bx}(2b\cos x + \sin x)}{4b^2+1} + c$$

or $$y^2 = \frac{2a(2b\cos x + \sin x)}{4b^2+1} + ce^{-2bx}$$

Q.12. Solve the differential equation

$$x^2 y \frac{dy}{dx} = xy^2 - e^{-\frac{1}{x^3}}$$

Sol. The equation can be written as

$$y\frac{dy}{dx} - \frac{1}{x} y^2 = -\frac{e^{-1}/x^3}{x^2}$$

Put $y^2 = z$

$$\therefore \quad 2y\frac{dy}{dx} = \frac{dz}{dx}$$

or $$y\frac{dy}{dx} = \frac{1}{2}\frac{dz}{dx}$$

$\therefore$ the equation is transformed to

$$\frac{1}{2}\frac{dz}{dx} - \frac{1}{x}z = -\frac{-e^{-1/x^3}}{x^2} \quad \text{or} \quad \frac{dz}{dx} - \frac{2}{x}z = -\frac{2e^{\frac{1}{x^3}}}{x^2}$$

which is linear differential equation in z

$\therefore$ I.F.$= e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$

$\therefore$ the solution is

$$\frac{z}{x^2} = \int \frac{-2e^{-\frac{1}{x^3}}}{x^2} \cdot \frac{1}{x^2}\, dx + \frac{c}{3}$$

or $$\frac{y^2}{x^2} = 2\int -e^{\frac{-1/x^3}{x^4}} .dx + \frac{c}{3}$$

Put $\frac{-1}{x^3} = t$

i.e. $-x^{-3} = t$

$3x^{-4}\, dx = dt$

or $x^{-4}\, dx = \frac{1}{3}\, dt$

or $\frac{dx}{x^4} = \frac{1}{3}$ dt

$\therefore$ the solution is

$$\frac{y^2}{x^2} = 2\int -e^t . \frac{1}{3} dt + \frac{c}{3}$$

$$= \frac{2}{3}\int -e^t dt + \frac{c}{3}$$

$$= -\frac{2}{3} e^t + \frac{c}{3}$$

or $y^2 = -\frac{2}{3} x^2 e^{-\frac{1}{x^3}} + \frac{c}{3}$

or $3y^2 + 2x^2 e^{-\frac{1}{x^3}} = c$

Q. 13. Solve the differential equation

$(y \log x - 1)\, y\, dx = x\, dy$

Sol. The equation can be written as

$$x \frac{dy}{dx} = y^2 \log x - y$$

or $x \frac{dy}{dx} + y = y^2 \log x$

or $y^{-2} \frac{dy}{dx} + \frac{y^{-1}}{x} = \frac{\log x}{x}$

Put $y^{-1} = z$

$\therefore \quad -y^{-2} \frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \quad y^{-2} \frac{dy}{dx} = -\frac{dz}{dx}$

$\therefore$ the equation is transformed to

$$-\frac{dz}{dx} + \frac{z}{x} = \frac{\log x}{x}$$

or $\frac{dz}{dx} - \frac{z}{x} = -\frac{\log x}{x}$

which is linear differential equation in z

$\therefore$ I.F$= e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

$\therefore$ the solution is

$$\frac{z}{x} = \int \frac{-\log x}{x} \times \frac{1}{x}\ dx$$

$$= -\int \log x \times \frac{1}{x^2} dx$$

$$= -\left[\log x\left(-\frac{1}{x}\right) - \int \frac{1}{x}\left(-\frac{1}{x}\right)dx\right]$$

$$= \frac{\log x}{x} + \int \frac{1}{x^2} dx$$

$$\therefore \quad \frac{y^{-1}}{x} = \frac{\log x}{x} - \frac{1}{x} + \text{c}$$

$$\therefore \quad \frac{1}{xy} = \frac{\log x}{x} - \frac{1}{x} + \text{c}$$

$\therefore$ $1 = y\,(cx + \log x - 1)$

or $y(cx + \log x - 1) = 1$

Q.14. Solve the differntial equation

$y(2xy + e^x)dx - e^x\,dy = 0$

Sol. The equation can be written as

$$e^x \frac{dy}{dx} = 2xy^2 + e^x.y$$

or $e^x \frac{dy}{dx} - e^x y = 2xy^2$

or $\frac{dy}{dx} - y = 2xe^{-x}y^2$

∴ $y^{-2}\frac{dy}{dx} - y^{-1} = 2x\ e^{-x}$

∴ Put $-y^{-1} = z$

∴ $y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$

∴ the equation is transformed into

$$\frac{dz}{dx} + z = 2x\ e^{-x}$$

which is linear different equation in z

∴ I.F.$= e^{\int dx} = e^x$

∴ the solution is

$$ze^x = \int 2x\ e^{-x}e^x dx + C$$

∴ $-y^{-1}e^x = \int 2x\ dx + C$

∴ $\frac{-e^x}{y} = x^2 + C$

or $e^x + y(x^2 + C) = 0$

-0-

Exact differential Equations

Recall. For a differentiable function *f(x, y),* the differential (total) is :

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

Definition. The expression

$$M(x, y)\,dx+N(x, y)\,dy \qquad (*)$$

is called an exact differential if there exists a function *f* of two variables *x* and *y* such that this expression equals the total differential *df*. That is, expression denoted by (*) is an exact differential if there exists a function *f* such that

$$M(x, y)\,dx+N(x, y)\,dy=df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

Comparing, we obtain

$$\frac{\partial f}{\partial x}=M(x, y),\ \frac{\partial f}{\partial y}=N(x, y)$$

Exact differential equation

Defintion. If $M(x, y)\,dx+N(x, y)\,dy$ is an exact differential then the differential equation $M(x, y)\,dx+N(x, y)\,dy=0$ is called an exact differential equation.

Note. The differential equation

$$y^2 dx+2xy\,dy=0$$

is an axact differential equation, since

$$y^2\,dx+2xy\,dy=df, \text{ where } f(x, y)=xy^2$$

Test to determine whether or not a given differential equation is exact. If so, method for finding the function $f(x, y)$.

Necessary and sufficient condition that the differential equation

M$(x, y)dx$+N$(x, y)\,dy$ =0 may be exact.

Necessary Condition

Suppose that the differential quation

$$M(x, y)\,dx+N(x, y)\,dy=0 \qquad \text{....(1)}$$

is exact, then there exist a function f of two variables x and y such that

$$M(x, y)\,dx+N(x, y)\,dy=df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy \qquad \text{....(2)}$$

Comparing we obtain

$$\frac{\partial f}{\partial x}=M,\ \frac{\partial f}{\partial y}=N \qquad \text{.....(3)}$$

From Calculus we know that the mixed second partial derivatives of f are equal
:

$$\frac{\partial^2 f}{\partial y dx}=\frac{\partial^2 f}{\partial x dy} \qquad \text{.....(4)}$$

provided both sides of (4) exist and are continuous.

Note the equation (4) when written in the form

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

yields

$$\frac{\partial M}{\partial y} = \frac{\partial N}{dx} \qquad .....(5)$$

which is the required necessary condition.

Suffcient condition

We shall now prove that (5) is also sufficient. This means that we must prove that there exists a function *f* such that

$$\frac{\partial f}{\partial x} = \mathrm{M}(x, y) \text{ and } \frac{\partial f}{\partial y} = \mathrm{N}(x, y)$$

i.e., conditions (3) are satisfied.

We can certainly find some $f(x, y)$ satisfying either of the above two conditions. Let us assume that *f* satisfies $\frac{\partial f}{dx} = \mathrm{M}(x, y)$ and then proceed to find *y*.

Now $\frac{\partial f}{dx} = \mathrm{M}(x, y)$

gives $f(x, y) = \int \mathrm{M}(x, y)\, dx + g(y)$

where *g(y)* is an *arbitrary function* of *y* and integration is to be performed by keeping y as constant.

Now if $\frac{\partial f}{dx} = \mathrm{N}(x, y)$ is to be *satisfied*, we must have

$$\mathrm{N}(x, y) = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int \mathrm{M}(x, y)\, dx + g'(y) \qquad ....(7)$$

Since g in equation (6) is a function of y only, the derviative $g'(y)$ in equation (7) must be independent of x. This can happen if

$$N(x, y) - \frac{\partial}{\partial y} \int M(x, y)\, dx$$

is independent of x Refer equation (8)).

We shall show that

$$\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx\right] = 0$$

Since $\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx\right]$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x \partial y}\int M(x, y)\, dx$$

$$= \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial y \partial x}\int M(x, y)\, dx \left(\Theta\ \frac{\partial^2}{\partial y \partial x} = \frac{\partial}{\partial x \partial y}\right)$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

=0 by virtue of (5)

Thus we may write equation (8) as

$$g(y) = \int\left[N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx\right]dy \qquad ...(9)$$

When we substitute this value of g(y) in equation (6) we obtain

$$f(x, y) = \int M(x,y)dx + \int \left[N(x,y) - \frac{\partial}{\partial y} \int M(x,y)dx \right] dy \quad .....(10)$$

This $f(x, y)$ thus satisfies both the conditions embodied in equation (3) and so $M(x, y)\,dx + N(x, y)dy = 0$ is exact.

Note. To use the test of exactness, the differential equation must be in the standard form viz.

$M(x, y)\,dx + N(x, y)\,dy = 0$

Working rule Equation (1) is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and in thus case, its **general solution** is given by $f(x, y) = c$, where f is given by equation 10.

Let us now discuss a few examples

Example 1. Test the equation

$e^y\,dx + (xe^y + 2y)\,dy = 0$

for **exactness**, and solve it if it is exact.

Sol. We have

$e^y\,dx + (xe^y + 2y)\,dy = 0$

Here $M(x, y) = e^y$ $\qquad$ $N(x, y) = xe^y + 2y$

$\therefore \quad \frac{\partial M}{\partial y} = e^y \qquad \frac{\partial N}{\partial x} = e^y$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$ and hence the given equation is exact. Its **general solution** is given by $f(x, y) = c$ where

$$f(x, y) = \int M(x,y)dx + \int \left[N(x,y) - \frac{\partial}{\partial y} \int M(x,y)dx \right] dy$$

$$=\int e^y dx + \int \left[(xe^y + 2y) - \frac{\partial}{\partial y} \int e^y dx \right] dy$$

$$=xe^y + = \int \left[\left(xe^y + 2y \right) - \frac{\partial}{\partial y} xe^y \right] dy \; dy$$

$$=xe^y + \int \left[(xe^y + 2y) - xe^y \right] dy$$

$$=xe^y + 2 \int y dy$$

$$=xe^y + y^2$$

$\therefore$ the required general solution is

$xe^y + y^2 = c$, c being arbitrary constant.

Aliter. We can write the given equation as

$(e^y dx + xe^y dx) + 2y\, dy = 0$

or $d(xe^y) + d(y^2) = 0$

or $d(xe^y + y^2) = 0$

This on integration gives $xe^y + y^2 = c$, c being arbitrary constant.

The above method of solution is called the 'Method of Grouping.'

Example 2. Solve the differential equation

$(2x^2y^3 + xy^2 + 3y)\, dx + (2x^3y^2 + x^2y + 3x)\, dy = 0$

Sol. We have

$(2x^2y^3 + xy^2 + 3y)\, dx + (2x^3y^2 + x^2y + 3x)\, dy = 0$ ....(1)

Here $M(x, y) = 2x^2y^3 + xy^2 + 3y$,

$$\frac{\partial M}{\partial y} = 6x^2y^2+2xy+3$$

$N(x, y)=2x^3y^2+x^2y+3x.$

$$\frac{\partial N}{\partial x} = 6x^2y^2+2xy+3$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ (1) is exact.

Its general solution is given by $f(x, y)=c$, where

$$f(x, y)=\int M(x, y)\, dx+\int\left[N(x,y)-\frac{\partial}{\partial y}\int M(x,y)dx\right]dy \quad (*)$$

Now $\int M(x, y)\, dx=\int (2x^2y^3+xy^2+3y)\, dx$

$$=2\frac{x^3}{3}y^3+\frac{x^2}{2}y^2+3xy,$$

$$\frac{\partial}{\partial y}\int M(x, y)\, dx=2x^3y^2+x^2y+3x,$$

$$-\frac{\partial}{\partial y}\int M(x, y)\, dx=(2x^3y^2+x^2y+3x)-(2x^3y^2+x^2y+3x)=0$$

With these values (*) becomes

$$f(x, y)=\frac{2}{3}x^3y^3+\frac{1}{2}x^2y^2+3xy$$

∴ the required general solution is

$\frac{2}{3}x^3y^3+\frac{1}{2}x^2y^2+3xy=c$, c being arbitiary constant.

Aliter. Method of Grouping

We write the given equation as

$$(2x^2y^3dx+2x^3y^2dy)+(xy^2dx+x^2ydy)+3(ydx+xdy)=0$$

or $$\frac{2}{3}(3x^2y^3dx+3x^3y^2dy)+\frac{1}{2}(2xy^2dx+2x^2y\, dy)+3(ydx+xdy)=0$$

or $$\frac{2}{3}d(x^3y^3)+\frac{1}{2}d(x^2y^2)+3\, d(xy)=0$$

or $$d\left(\frac{2}{3}x^3y^3+\frac{1}{2}x^2y^2+3xy\right)=0$$

This on integration gives $\frac{2}{3}x^3y^3+\frac{1}{2}x^2y^2+3xy = c$

c being arbitrary constant.

Integration Factors (I.F.)

Consider the differential equation

$$\frac{dy}{dx}+Py=Q \qquad .....(1)$$

where P, Q are functions of x

We write the above differential equation in the form

$$(Q-Py)\, dx-dy=0$$

We have, $M(x, y)=Q-Py$ $\qquad N(x, y)=-1$

$$\therefore \quad \frac{\partial M}{\partial y}=-P \qquad \frac{\partial N}{\partial x}=0$$

Therefore, equation (1) is not exact unless P=0.

However, if we multiply (1) by $e^{\int Pdx}$, if is transformed into the equivalent equation.

$$e^{\int Pdx}\left(\frac{dy}{dx}+Py\right)=Qe^{\int Pdx}$$

which further can be written as

$$\frac{d}{dx}\left(ye^{\int Pdx}\right)=Qe^{\int Pdx}$$

whose solution, on integration, yields

$ye^{\int Pdx}=c+\int Qe^{\int Pdx}\,dx$, c being arbitrary constant.

We call $e^{\int Pdx}$ an integrating factor (I.F.).

In general, we have the following definition :

Definition. If the differential equation

$M(x, y)\,dx+N(x, y)\,dy=0$

is not exact but the differential equation

$\mu(x, y)\,M(x, y)\,dx+\mu(x, y)\,N(x, y)\,dy=0$

is exact, then $\mu(x, y)$ is called an integration factor (I.F.)

Rules for finding integrating factors (IF.)

1. If the equation $M\,dx+N\,dy=0$ is of the form $f_1(x)\,\phi_1(y)\,dx+f_2(x)\,\phi_2(y)dy=0$, then the integrating factor

$$\mu(x, y)=\frac{1}{f_2(x)\phi_1(y)}$$

2. If the equation is of the form $M\,dx+N\,dy=0$, where M and N are homogenous functions of the same degree, then the integrating factor

$$\mu(x, y)=\frac{1}{Mx+Ny}$$

provided $Mx+Ny\neq 0$.

3. If the equation M dx+N dy=0 is of the form

$y f_1(xy)\, dx+x f_2(xy)\, dy=0$, then the integrating factor.

$$\mu(x, y)=\frac{1}{Mx-Ny}, \text{ provided } Mx-Ny\neq 0$$

4. If the equation is of the form $Mdx+Ndy=0$ and $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=g(x)$, then the integrating factor

$$\mu(x)=e^{\int g(x)dx}$$

5. If the equation is of the form M dx+N dy=0 and $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{M}=h(y)$, then the integrating factor

$$\mu(y)=e^{\int h(y)dy}$$

6. If the equation of the form $Mdx+Ndy=0$ where M and N are simple algebraic functions we shall try an expression of the form $\mu(x, y)=x^m y^n$; m, n are to be determined.

Example 3. Solve the differential equation

$$(2xy+y^2)dx+(2x^2+3xy)dy=0$$

Sol. The given differential equation is

$$(2xy+y^2)dx+(2x^2+3xy)dy=0 \qquad ...(1)$$

Here $M(x, y)=2xy+y^2$ $\qquad$ $N(x, y)=2x^2+3xy$

$\frac{\partial M}{\partial y}$=2$x$+2y $\frac{\partial N}{\partial x}$=4$x$+3$y$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ (1) is not exact.

We first make (1) to be exact differential equation by finding the integrating factor (I.F.)

We have

$$\frac{-\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = -\frac{2(x+y)-(4x+3y)}{y(2x+y)}$$

$$=-\frac{-2x-y}{y(2x+y)}$$

$=\frac{1}{y}$, a function of y alone

∴ I.F. $=e^{\int 1/y\, dy}=e^{\log y}=y$

We now multiply (1) throughout by y to get

$(2xy^2+y^3)\,dx+(2x^2y+3xy^2)dy=0$

or $(y^3dx+3xy^2dy)+(2xy^2dx+2x^2ydy)=0$ (By Grouping)

or $d(xy^3)+d(x^2y^2)=0$

or $d(xy^3+x^2y^2)=0$

Integration yields, $xy^3+x^2y^2=c$,

c being arbitrary constant.

Example 4. Solve the differential equation

$$(xy^2+2x^2y^3)dx+(x^2y-x^3y^2)\ dy=0$$

Sol. The given differential equation is

$$(xy^2+2x^2y^3)\ dx+(x^2y-x^3y^2)\ dy=0 \qquad \text{....(1)}$$

Here $M(x, y)=xy^2+2x^2y^3$ $\qquad N(x, y)=x^2y-x^3y^2$

$$\frac{\partial M}{\partial y}=2xy+x^2y^2 \qquad \frac{\partial N}{\partial x}=2xy-3x^2y^2$$

Since $\qquad \dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$

$\therefore$ (1) is not exact,

We first make (1) to be exact differential equation by finding the integrating factor (I.F.)

Note that $M(x, y)$ and $N(x, y)$ are simple algebraic functions so we take

$$\mu\ (x, y)=x^my^n$$

so that

$$x^my^n(xy^2+2x^2y^2)dx+x^my^n(x^2y-x^3y^2)dy=0 \qquad \text{...(2)}$$

is exact. The criterion for exactness is

$$\frac{\partial}{\partial y}\left[x^my^n\left(xy^2+2x^2y^3\right)\right]=\frac{\partial}{\partial x}\left[(x^my^n(x^2y-x^3y^2)\right]$$

or $$\frac{\partial}{\partial y}\left[x^{m+1}y^{n+2}+2x^{m+2}y^{n+3}\right]=\frac{\partial}{\partial x}\left[(x^{m+2}y^{n+1}-x^{m+3}y^{n+1}\right]$$

or $$(n+2)x^{m+1}y^{n+1}+2(n+3)\ x^{m+2}\ y^{n+2}$$

$$=(m+2)x^{m+1}y^{n+1}-(m+3)x^{m+2}y^{n+2}$$

Comparing coefficients, we have

n+2=m+2 ....(i) 2(n+3)=–(m+3) ....(ii)

Solving (i) and (ii), we obtain m=n=–3

∴ yμ(x,)= $\frac{1}{x^3y^3}$

Multiplying (1) throughout by $\frac{1}{x^3y^3}$, we obtain

$$\left(\frac{1}{xy^2}+\frac{2}{x}\right)dx+\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0$$

or $$\left(\frac{1}{x^2y}dx+\frac{1}{xy^2}dy\right)+\frac{2}{x}dx\frac{1}{y}dy=0$$

or $$\frac{ydx+xdy}{(xy)^2}+\frac{2}{x}dx\frac{1}{y}dy=0$$

or $$\frac{d(xy)}{(xy)^2}+d(2\log x)–d(\log y)=0$$

or $$–d\left(\frac{1}{xy}\right)+d(\log x^2)–d(\log y)=0$$

or $$d\left[\log\left(\frac{x^2}{y}\right)\right]-d\left(\frac{1}{xy}\right)=0$$

or $\log \frac{x^2}{cy} = \frac{1}{xy}$, c being arbitrary constant.

Important Note. Multiplication of a non-exact differential equation by an integrating factor thus transforms the non-exact equation into an exact one. However, during the process there may result either (1) the loss of (one or more) solutions of the original equation or (2) the gain of (one or more) functions which are solutions of the new equation but not of the original or (3) both these phenomena. Thus it is to be remembered that whenever we transform a non-exact equation into an exact one (by multiplication) by an integrating factor, we should check carefully to determine whether any solutions may have been lost or gained.

Example 5. Solve the differential equation

$(x-4)\, y^4\, dx - x^3\, (y^2-3)\, dy = 0.$

Solution. The given differential equation is

$(x-y^4-4y^4)\, dx + (3x^3 - x^3y^2)\, dy = 0$ ...(1)

Here $m(x, y) = xy^4 - 4y^3$ $\quad$ $N(x, y) = 3x^3 - x^3y^2$

$\frac{\partial M}{\partial y} = 4xy^3 - 16y^3$ $\quad$ $\frac{\partial N}{\partial x} = 9x^2 - 3x^2y^2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ (1) is not exact.

However, we can write the given equation as

$$\frac{x-4}{x^3}dx - \frac{y^2-3}{y^4}\,dy = 0$$

or $\quad \frac{1}{x^2}dx - \frac{4}{x^3}dx - \frac{1}{y^2}dy + \frac{3}{y^4}\,\mathrm{d}y=0$

or $\quad d(-x^{-1})+d(2x^{-2})+d(y^{-1})+d(-y^{-3})=0$

or $\quad d(2x^{-2}+y^{-1}x^{-1}-y^{-3})=0$

Integration yields

$2x^{-2}+y^{-1}-x^{-1}-y^{-3}=c$, c being arbitrary constant.

Note. The solution $y=0$ of the original equation has been lost during the 'process'.

EXERCISES

Verify that the differential equations (1—8) are exact. Solve them completely.

1. $(ax+hy+g)\,dx+(hx+by+f)\,dy=0$
2. $(2y-2)dy+(2x+3)dx=0$
3. $x(1+y^2)dx+y(1+x^2)dy=0$
4. $(2xy^3+y\cos x)dx+(3x^2y^2+\sin x)dy=0$
5. $(2xy^2+2y)\,dx+(2x^2y+2x)dy=0$
6. $(e^x\sin y-2y\sin x)\,dx+(e^x\cos y+2\cos x)dy=0$
7. $(1+e^{x/y})\,dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$
8. $(ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x)\,dx+(xe^{xy}\cos 2x-3)\,dy=0$
9. Show that $\mu(x, y)=xy$ is an integrating factor (I.F.) of the differential equation

   $$\left(\frac{2x+y+3}{x}\right)dx+\left(\frac{x+2y+3}{y}\right)dy=0$$

   and hence find its general solution.
10. Find an integrating factor (I.F.) for the equation

$(3xy+y^2)dx+(x^2+xy)$ dy=-0

and hence find its general solution.

11. Find an integrating factor (I.F.) for the equation

$y\ dx+(x-y \sin y)\ dy=0$

and hence its general solution.

12. Show that $\mu(x,y)=\frac{1}{xy^2}$ is an **integrating factor** of the differential equation.

$(y^2+xy)\ dx-x^2\ dy=0$

and hence find its general solution.

ANSWERS

1. $ax^2+2hxy+by^2+2gx+2fy+c=0$
2. $x^2+3x+y^2-2y=c$
3. $(1+x^2)(1+y^2)=c$
4. $x^2y^3+y \sin x=c$
5. $x^2y^2+2xy=c$
6. $e^x \sin y+2y \cos x=c$
7. $x+y\ e^{x/y}=c$
8. $e^{xy} \cos 2x+x^2-3y=c$
9. $x^2y+xy^2+3xy=c$
10. $\mu(x)=x, x^3y+\frac{1}{2}x^2y^2=c$
11. $\mu(y)=y, xy+y \cos y-\sin y=c$
12. $\log x+\frac{x}{y}=c$

Equations of the First Order but not of the First degree

A first order differential equation not solved for the derivative is of the form

$$F(x,y,p)=0, p=\frac{dy}{dx}.$$

Equations solvable for p

Let the equation of the nth degree in p be

$$f(x, y, p)=0.$$

Since it is solvable for p,

$\therefore (p-m_1)(p-m_2)....(p-m_n)=0,$

where m_1 m_2,......, m_n are funcitons of x and y.

Solving, we have

$p-m_1=0 \; |p-m_2=0| \; ....| \; p-m_n=0.$

Let the solutions be

$f_1(x, y, c_1)=0, f_2(x, y, c_2)=0, ..., f_n(x, y, c_n)=0$

Since the given euqation is of the first order.

$\therefore$ it can't have more than one arbitrary constant in it solution.

For this let $c_1 = c_2 = ... = c_n = c_1$ *say.*

So the above solutions become

$f_1(x,y,c)=0, f_2(x,y,c)=0, \ldots, f_n(x,y,c)=0.$

Hence the joint solution is

$f_1(x,y,c)\ f_2(x,y,c)\ \ldots f_n(x,y,c)=0.$

Note. The joint solution is also called the complete primitive.

Rule to solve :

(i) Resolve the given equation into linear factors of p.

(ii) Equate each factor to zero, which will give a differential equation.

(iii) Solve these equations to get the required solution.

Example 1. Solve the differential equation

$$p^2-5p+6=0$$

Sol. The given equation is $p^2-5p+6=0$

Solving, we obtain $(p-2)(p-3)=0$

$\therefore\ p=2, 3$

when $p=2$, we have $\frac{dy}{dx}=2$ or $y=2x+c$

when $p=3$, we have $\frac{dy}{dx}=3$ or $y=3x+c$.

Hence the complete primitive is

$$(y-2x-c)(y-3x-c)=0,$$

where c is any arbitrary constant.

Example 2. Solve the differential equation

$$4\,p^2x = (3x-a)^2$$

Sol. The given differential equation is

$$4p^2\,x=(3x-a\,)^2$$

Solving, we obtain $p=\dfrac{3x-a}{2\sqrt{x}}$

or $\dfrac{dy}{dx}=\dfrac{3x-a}{2\sqrt{x}}$

or $dy=\dfrac{3}{2}\sqrt{x}-\dfrac{1}{x}a/\sqrt{x}.$

Integrating, we obtain

$y+c=x^{3/2}-a\;x^{1/2}$

or $(y+c)=x^{1/2}\,(x\text{-}a)$

or $(y+c)^2=x(x-a)^2$

which is the required primitive.

Example 3. Solve the differential equation

$x^2p^2-2xyp+2y^2-x^2=0.$

Sol. The given differential equation is

$x^2p^2-2xyp+(2y^2-x^2)=0$

Solving for p, we obtain

$$p=\frac{2xy\underline{+}\sqrt{4x^2y^2-4x^2(2y^2-x^2)}}{2x^2}$$

$$=\frac{2xy\pm 2x\sqrt{x^2-y^2}}{2x^2}$$

$$=\frac{y}{x}\pm\sqrt{1-\left(\frac{y}{x}\right)^2}$$

or $$\frac{dy}{dx}=\frac{y}{x}\pm\sqrt{1-\left(\frac{y}{x}\right)^2} \qquad .....(1)$$

which is homogeneous in x and y.

Put $y=tx \quad \therefore \frac{dy}{dx}=t+x\frac{dt}{dx}$.

With these values (1) becomes

$$t+x\frac{dt}{dx}=t\pm\sqrt{1-t^2}$$

or $$\frac{dt}{\sqrt{1-t^2}}=\pm\frac{dx}{x}$$

Integrating, we obtain

$\sin t = \log x+c \qquad\qquad sin\ t=-\log x+c$

or $\sin^{-1}\frac{y}{x}=\log cx \qquad sin^{-1}\frac{y}{x}=-\log cx$

$\therefore$ complete primitive is

$$\sin^{-1}\frac{y}{x}=\pm\log cx$$

EXERCISE 1

Solve the following differential equations :

(a) $p^2-7p+12=0$

(b) $p^2+p-6=0$

(c) $p^2+2px-3x^2=0$

(d) $p^2-x^5=0$

(e) $xp^2+(y-x)\,p-y=0$

(f) $(px+x+y)(p+x+y)(p+2x)=0$

(g) $p^2-2p\cos hx+1=0$

(h) $p^2+2py\cot x-y^2=0$

(i) $xyp^2-(x^2+y^2)p+xy=0$

(j) $\left(1-y^2+\frac{y^4}{x^2}\right)p^2-2\frac{y}{x}p+\frac{y^2}{x^2}=0$

ANSWERS

(a) $(4x-y+c)(3x-y+c)=0$

(b) $(2x-y+c)(3x+y-c)=0$

(c) $(2y-x^2-c)(2y+3x^2-c)=0$

(d) $49(y-c)^2=4x^7$

(e) $(y-x-c)(xy-c)=0$

(f) $(2xy+x^2-c)(y+x-1-ce^{-x})(x^2+y-c)=0$

(g) $(y-e^x-c)(y+e^{-x}-c)=0$

(h) $[y(1+\cos x)-c)+[y(1-\cos x)-c]=0$

(i) $(y^2-x^2-c)(y-cx)=0$

(j) $\left[\log\left(\frac{x}{y}+\frac{\sqrt{x^2-y^2}}{y}\right)+y-c\right]\left[\log\left(\frac{x}{y}+\frac{\sqrt{x^2-y^2}}{y}\right)-y-c\right]=0$

Clairaut's form and equations reducible to Clairaut's form:

Definition. A differential equation of the first order but not of the first degree is an equation of the form

$$Q_o P^{n} + Q_1 P^{n-1} + Q_2 P^{n-2} + ... + Q_{n-1} P + Q_n = 0$$

where $P = \frac{dy}{dx}$ and $Q_0, Q_1, Q_2........Q_n$ are functions of *x and y* $(n \geq 2), (n \in N)$.

Note. It is conversion to denote $\frac{dy}{dx}$ by p in equations of this type.

Type I. Equations solvable for y.

If the equation F $(x, y, p) = 0$ is solvable for y, we can express y explicity in terms of x and p. Thus, an equation solvable for y can be put as

$$y = \phi(x, p) \qquad ...(i)$$

Differentiating (i) w.r.t. x., we get

$$\frac{dy}{dx} = p = F\left\{x, p, \frac{dp}{dx}\right\} \qquad ...(ii)$$

which is a differential equation involving two variables x and p. Let its solution be

$$\phi(x, p, c) = 0 \qquad ..(iii)$$

Eliminating p between (i) and (iii), we get the required solution of (i)

If p cannot be easily eliminated, then we solve the equations (i) and (iii) for x and y in terms of p. Then the parametric equations

$$x=f_1(p,c)$$

$y=f_2(p,c)$, p being the parameter

These equations together constitute the solution of (i) Clairaut's Equation

OR

Special case of Type I.

Definition. A differential equation of the form

$$y=px+f(p)$$

is known as Clairaut's Equation, where

$$p=\frac{dy}{dx}$$

Article. To prove that the solution of the equation

$y=px+f(p)$ is

$$y=ex+f(c)$$

Sol. The given equation is

$$y=px+f(p) \qquad ..(i)$$

It is solvable for y

Differentiating (i) w.r.t.x, we have

$$\frac{dy}{dx}=P+x\frac{dp}{dx}+f'(p)\frac{dp}{dx}$$

$$\Rightarrow \qquad p=p+x\frac{dp}{dx}+f'(p)\frac{dp}{dx}$$

$$\Rightarrow \qquad x\frac{dp}{dx}+f'(p)\frac{dp}{dx}=0$$

$\Rightarrow \qquad \frac{dp}{dx}\left[x + f(p)\right] = 0 \qquad$ ...(i)

Cancelling the factor $x+f(p)$ which does not involve $\frac{dp}{dx}$, we have

$$\frac{dp}{dx}=0$$

Integrating, we have

$p=c \qquad$ ...*(ii)*

Eliminating p between (i) and and (ii), the required solution of (i) is

$$y=cx+f(c)$$

Also from (I), we have

$x+f(p) = 0 \qquad$ ..*(iii)*

Eliminating p from (i) and (iii) i.e. solving (i) and (iii), we obtain another solution of (i). This solution of (i) is free of any constant and is called singular solution.

Note. that (iii) can be obtained by differentiating (i) partially w.r.t. p.

Q.1. Solve the equation

$p=xp+\frac{a}{p} \qquad$ ..*(i)*

Differentiating the given equation w.r.t.x, we have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$$

$\Rightarrow \qquad p=p+x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}$

$$\Rightarrow \qquad \left(x - \frac{a}{p^2}\right)\frac{dp}{dx} = 0$$

Hence either $\frac{dp}{dx}=0$

$\therefore \qquad p=c$

$\therefore$ the complete solution is

$$y=cx+\frac{a}{c}$$

Also $\quad x - \frac{a}{p^2} = 0 \qquad ..(2)$

Eliminating p between (1) and (2), we have

From (2) $\quad P^2=\frac{a}{x} \quad \Rightarrow \quad p = \sqrt{\frac{a}{x}}$

Substituting in = n (1) we have

$$y = x\sqrt{\frac{a}{x}} + \frac{a}{\sqrt{\frac{a}{x}}}$$

$$= \sqrt{ax} + \sqrt{ax}$$

$$\therefore \qquad y = 2\sqrt{ax}$$

Squaring both sides, we have

$$y^2=4ax$$

which is the singular solution.

The parabola y^2=4ax is evidently the envelope of the tangents.

$$y=cx+\frac{a}{c}$$

Q. 2. Solve the equation

$$y=px-p^2$$

Find its singular solution also.

Sol. The given equation is

$$y=px-p^2$$

Differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx}=p+x\frac{dp}{dx}-2p\frac{dp}{dx}$$

$\Rightarrow$ $$p=p+\frac{dp}{dx}(x-2p)=0$$

$\Rightarrow$ $$\frac{dp}{dx}(x-2p)=0$$

Either $$\frac{dp}{dx}=0$$

$\Rightarrow$ $$p=c$$

$\therefore$ the complete solution is

$$y=cx-c^2$$

Also $$x-2p=0$$

$\Rightarrow$ $$2p=x$$

$\Rightarrow$ $$p=\frac{x}{2}$$

Putting this in the given equation, we have

$$y=\frac{x}{2}.x-\left(\frac{x}{2}\right)^2$$

$$=\frac{x^2}{2}-\frac{x^2}{4}=\frac{x^2}{4}$$

$\Rightarrow \quad x^2-4y=0$

which is the required singular solution.

Q.3. Solve the differential equation

$$y = px+\sqrt{a^2p^2+b^2},$$

$$p=\frac{dy}{dx}$$

Find its singular solution also.

Sol. The given $n =$ is

$$y=px+\sqrt{a^2p^2+b^2}$$

Differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx}=p+x\frac{dp}{dx}+\frac{1}{2}\frac{1}{\sqrt{a^2p^2+b^2}}.2a^2p\frac{dp}{dx}=0$$

$$\Rightarrow \quad p=p+\frac{dp}{dx}\left[x+\frac{a^2p}{\sqrt{a^2p^2+b^2}}\right]=0$$

$$\Rightarrow \qquad \frac{dp}{dx}\left[x + \frac{a^2 p}{\sqrt{a^2 p^2 + b^2}}\right] = 0$$

Either $\frac{dp}{dx}=0 \quad \Rightarrow \qquad p=c$

$\therefore$ the complete solution is

$$y = cx + \sqrt{a^2 c^2 + b^2}$$

or $\qquad x + \frac{a^2 p}{\sqrt{a^2 p^2 + b^2}} = 0$

$\therefore \qquad \sqrt{a^2 p^2 + b^2}\ \ x = -a^2 p \qquad \Rightarrow \sqrt{a^2 p^2 + b^2} = -\frac{a^2 p}{x}$

or $\qquad x^2 (a^2p^2+b^2) = a^4p^2$

or $\qquad a^2p^2 (x^2-a^2) = - b^2x^2$

$\Rightarrow \qquad p^2 = \frac{b^2 x^2}{a^2\left(a^2 - x^2\right)} \qquad \Rightarrow \qquad p = \frac{bx}{a\sqrt{a^2 - x^2}}$

Substituting in the given equation, we have

$$y = \frac{bx^2}{a\sqrt{a^2 - x^2}} - \frac{a^2 p}{x}$$

$$y = \frac{bx^2}{a\sqrt{a^2 - x^2}} - \frac{a^2}{x}\frac{bx}{a\sqrt{a^2 - x^2}}$$

$$= \frac{b}{\sqrt{a^2 - x^2}}\left[\frac{x^2}{a} - a\right]$$

$$\frac{-b\left(a^2-x^2\right)}{a\sqrt{a^2-x^2}}$$

$$\therefore \quad y=\frac{-b}{a}\sqrt{a^2-x^2}$$

Squaring both sides we have

$$y^2=\frac{b^2}{a^2}\ (a^2-x^2)$$

$$\Rightarrow \qquad \frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$\Rightarrow \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

which is the singular solution.

Q. 4. Solve the differential equation

$$y=px+\sqrt{a^2p^2-b^2},$$

$$p=\frac{dy}{dx}$$

Find its singular solution also.

Sol. The given = *n* is

$$y=px+\sqrt{a^2p^2-b^2}$$

Changing b^2 to $-b^2$ in Q. 3, we get the complete solution as

$$y=cx+\sqrt{a^2c^2-b^2}$$

and the singular solution is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Q. 5. Solve the differential equation

$y = px + p^3$

Find its singular solution also

Sol. The given equation is

$y = px + p^3$

Differentiating both sides w.r.t.x., we have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + 3p^2\frac{dp}{dx}$$

$$\Rightarrow \qquad p = p + x\frac{dp}{dx} + 3p^2\frac{dp}{dx}$$

$$\Rightarrow \qquad \frac{dp}{dx}(x + 3p^2) = 0$$

Either $\frac{dp}{dx} = 0 \qquad \Rightarrow \qquad p = c$

$\therefore$ the complete solution is

$y = cx + c^3$

or $x + 3p^2 = 0$

$\Rightarrow \qquad 3p^2 = -x$

$\Rightarrow \qquad p^2 = -\frac{x}{3}$

$\Rightarrow \qquad p=\left(\frac{-x}{3}\right)^{\frac{1}{2}}$

Substituting this value of p in the given = n, we have

$$y=x\left(\frac{-x}{3}\right)^{\frac{1}{2}}+\left(\frac{-x}{3}\right)^{3/2}$$

$\Rightarrow \qquad y=-3\left(\frac{-x}{3}\right)^{3/2}+\left(\frac{-x}{3}\right)^{3/2}$

$\therefore \qquad y=-2\left(\frac{-x}{3}\right)^{3/2}$

Squaring both sides, we have

$$y^2=4\left(\frac{-x^3}{27}\right)$$

or $\qquad 27y^2=-4x^3$

$\Rightarrow \qquad 4x^3+27y^2=0$

which is the singular solution of given equation

Q. 6. Solve the differential equation

$$y=px+p-p^2,\ p=\frac{dy}{dx}$$

and find its singular solution.

Sol. The given equation is

$$y=px+p-p^2$$

Differentiating both sides w.r.t.x, we have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + \frac{dp}{dx} - 2p\frac{dp}{dx}$$

$\Rightarrow \quad P=P+(x+1-2p)\frac{dp}{dx}$

$\Rightarrow \quad (x+1-2p)\frac{p}{dx}=0$

Either $\quad \frac{dp}{dx}=0 \quad \Rightarrow \quad p=c$

$\therefore$ the complete solution is

$$y=cx+c-c^2$$

or $\quad x+1-2p=0$

$\Rightarrow \quad 2p=x+1$

$\Rightarrow \quad p=\frac{x+1}{2}$

Substituting for *p* in the given equation, we have

$$y = \frac{x+1}{2}x + \frac{x+1}{2} - \left(\frac{x+1}{2}\right)^2$$

$\Rightarrow \quad 4y=2x^2+2x+2x+2-x^2-2x-1$

$\Rightarrow \quad 4y=x^2+2x+1$

$\Rightarrow \quad 4y=(x+1)^2$

which is the singular solution.

Q. 7. Solve the differential equation

$$y = px + a\sqrt{1+p^2}\,,\ p = \frac{dy}{ax}$$

Find its singular solution also

Sol. The given equation is

$$y = px + a\sqrt{1+p^2}$$

Differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + a\frac{1}{2\sqrt{1+p^2}} \times 2P\frac{dp}{dx}$$

$\Rightarrow$ $$p = p + \frac{dp}{dx}\left(x + \frac{ap}{\sqrt{1+p^2}}\right) = 0$$

$\Rightarrow$ $$\frac{dp}{dx}\left(x + \frac{ap}{\sqrt{1+p^2}}\right) = 0$$

Either $\dfrac{dp}{dx} = 0 \quad \Rightarrow \quad p = c$

$\therefore$ the complete solution is

$$y = cx + a\sqrt{1+c^2}$$

or $$x + \frac{ap}{\sqrt{1+p^2}} = 0$$

$\Rightarrow$ $$x = -\frac{ap}{\sqrt{1+p^2}}$$

$\Rightarrow \quad x\sqrt{1+p^2} = -ap$

$\Rightarrow \quad x^2(1+p^2)=a^2p^2$

$\Rightarrow \quad x^2+x^2p^2-a^2p^2=0$

$\Rightarrow \quad (a^2-x^2) \quad p^2=x^2$

$\Rightarrow \quad p^2=\frac{x^2}{a^2-x^2}$

$\Rightarrow \quad p=\frac{\pm x}{\sqrt{a^2-x^2}} = \frac{-x}{\sqrt{a^2-x^2}}$, taking –ve sign

Substituting this value of *p* in the given equation, we have

$$y = -x\frac{x}{\sqrt{a^2-x^2}} + a\sqrt{1+\frac{x^2}{a^2-x^2}}$$

$$= \frac{-x^2+a^2}{\sqrt{a^2-x^2}}$$

$\therefore \quad y = \sqrt{a^2-x^2}$

Squaring both sides, we have

$y^2=a^2-x^2$

$\Rightarrow \quad x^2+y^2=a^2$

which is the singular solution.

Q.8. Solve the differential equation

$$\sin(px-y) = p, \; p = \frac{dy}{dx}$$

Find its singular solution also

The given equation is

$\sin(px-y)=p$

$\Rightarrow \quad px-y=\sin^{-1}p$

$\Rightarrow \quad y=px-\sin^{-1}p$

Differentiating both sides w.r.t.x, we have

$$\frac{dy}{dx}=p+x\frac{dp}{dx}-\frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$$

$$\Rightarrow \quad p=p+\frac{dp}{dx}\left(x-\frac{1}{\sqrt{1-p^2}}\right)$$

$$\Rightarrow \quad \frac{dp}{dx}\left(x-\frac{1}{\sqrt{1-p^2}}\right)=0$$

Either $\frac{dp}{dx}=0 \quad \Rightarrow \quad p=c$

$\therefore$ the complete solution is

$y=cx-\sin^{-1}c$

or $\quad x-\frac{1}{\sqrt{1-p^2}}=0$

$$\Rightarrow \quad \sqrt{1-p^2}=\frac{1}{x}$$

$$\Rightarrow \quad 1-p^2=\frac{1}{x^2}$$

$\Rightarrow \quad p^2 = 1 - \frac{1}{x^2}$

$\Rightarrow \quad p = \sqrt{1 - \frac{1}{x^2}}$

$\therefore$ Substituting for p in the given equation, we have

$$y = x\sqrt{1 - \frac{1}{x^2}} - \sin^{-1}\sqrt{1 - \frac{1}{x^2}}$$

$\Rightarrow \quad y = \sqrt{x^2 - 1} - \sin^{-1}\sqrt{1 - \frac{1}{x^2}}$

Q. 9. Solve the differential equation

$$p^3 - 3xp + 3y = 0, \ p = \frac{dy}{dx}$$

Find its singular solution also.

Sol. The given = n is

$$p^3 - 3xp + 3y = 0$$

$\Rightarrow \quad 3y = 3xp - p^3$

$\Rightarrow \quad y = px - \frac{p^3}{3}$

Differentiating both sides w.r.t.x, we have

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{1}{3}.3p^2\frac{dp}{dx}$$

$\Rightarrow \quad p=p+\frac{dp}{dx}(x-p^2)=0$

$\Rightarrow \quad \frac{dp}{dx}(x-p^2)=0$

Either $\frac{dp}{dx}=0 \quad \Rightarrow \quad p=c$

Substituting this value in = n (1), we have the complete solution as

$$y=cx-\frac{1}{3}c^3$$

or $\quad x-p^2=0$

$\Rightarrow \quad p^2=x$

$\Rightarrow \quad p=\sqrt{x}$

Substituting this value of p *in* = n (i), we have

$$y=x\sqrt{x}-\frac{x^{3/2}}{3}$$

$$=x^{3/2}-\frac{x^{3/2}}{3}$$

$$=\frac{2x^{3/2}}{3}$$

$\Rightarrow \quad 3y=2x^{3/2}$

Squaring both sides, we have

$$9y^2=4x^3$$

which is the singular solution.

Type II : Equations reducible to Clairaut's form by transformation.

Q. 10. Solve the equatiion

$$y = 2px+y^2p^3$$

Sol. Multiplying = *n* (given) both sides by *y,* we have

$$y^2=2ypx+y^3p^3$$

$$\Rightarrow \quad y^2=2ypx+(yp)^3 \quad .....(i)$$

Put $y^2=z$

$$\Rightarrow \quad 2y\frac{dy}{dx}=\frac{dz}{dx}$$

$$\Rightarrow \quad 2yp=\frac{dz}{dx} \quad \Rightarrow \quad yp=\frac{1}{2}\frac{dz}{dx}$$

Substituting these values in = *n* (i), we have

$$z=x\frac{dz}{dx}+\left(\frac{1}{2}\frac{dz}{dx}\right)^3$$

$$\Rightarrow \quad z=x\frac{dz}{dx}+\frac{1}{8}\left(\frac{dz}{dx}\right)^3$$

which is of Clairaut's form.

Hence the solution is

$$z=cx+\frac{1}{8}c^3$$

or $$y^2=cx+\frac{1}{8}c^3$$

Q. 11. Solve the differential equation

$$(px-y)(py+x)=h^2p$$

Sol. Put $x^2=u,\ y^2=v$

$$\therefore \qquad 2x=\frac{du}{dx}, 2y\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow 2yp=\frac{dv}{dx}$$

$$\therefore \qquad \frac{dv}{du}=\frac{dv}{dx}\bigg|\frac{du}{dx}=\frac{2yp}{2x}=\frac{yp}{x} \qquad \text{....(1)}$$

The given = n may be written as

$$y\left(px.\frac{x}{x}-y\right)\left(\frac{py}{x}+1\right)=h^2 p\frac{y}{x}$$

$$\Rightarrow \qquad \left[\frac{py}{x}\left(x^2\right)-y^2\right]\left(\frac{py}{x}+1\right)=h^2\frac{py}{x}$$

Substituting for $\frac{py}{x}$ as $\frac{dv}{du}$ from (1), we have

$$\left(u\frac{dv}{du}-v\right)\left(\frac{dv}{du}+1\right)=h^2\frac{dv}{du}$$

$$\text{or} \qquad u\frac{dv}{du}-v=\frac{h^2\frac{dv}{du}}{1+\frac{dv}{du}}$$

$$\Rightarrow \qquad v=u\frac{dv}{du}=\frac{h^2\frac{dv}{du}}{1+\frac{dv}{du}}$$

which is of Clairaut's form.

Hence the solution is

$$v=uc-\frac{h^2c}{1+c}$$

or $\quad y^2=x^2c-\frac{h^2c}{1+c}$

$\Rightarrow \quad y^2-cx^2+\frac{ch^2}{1+c}=0$

Q. 12. Solve the differential equation

$$x^2(y-px)=yp^2$$

Sol. In the given equation

Put $\quad x^2=u$ and $y^2=v$

$\therefore \quad 2xdx=du$ and $2y\,dy=dv$

$\therefore \quad \frac{2y\,dy}{2x\,dx}=\frac{dv}{du}$

$\Rightarrow \quad \frac{y}{x}\frac{dy}{dx}=\frac{dv}{du}$

$\Rightarrow \quad \frac{y}{x}p=\frac{dv}{du}=p,$ say

or $\quad p=\frac{x}{y}p^2$

$\therefore$ the given equation becomes

$$x^2\left(y-\frac{x^2}{y}p\right)=y.\frac{x^2}{y^2}p^2$$

$\Rightarrow$ $(y^2-x^2\ P)\dfrac{x^2}{y}=\dfrac{x^2}{y}P^2$

$\Rightarrow$ $(y^2-x^2\ P)\ =\ P^2$

$\Rightarrow$ $y^2=x^2P+P^2$

$\Rightarrow$ $\upsilon=Pu+P^2$

which is of Clairaut's form.

$\therefore$ the solution is

$v=cu+c^2$

or $y^2=cx^2+c^2$

Q. 13. Solve the differential equation

$$(y+xp)^2=x^2p$$

Sol. **Put** $\boldsymbol{xy=z}$

$\therefore$ $x\dfrac{dy}{dx}+y=\dfrac{dz}{dx}=P\ (say)$

$\Rightarrow$ $xp+y=P$

$\therefore$ The given equation becomes

$P^2=x(xp)=x(P-y)$

$\Rightarrow$ $P^2=Px-xy$

$Px-z$

$\Rightarrow$ $z=Px-P^2$

which is of Clairaut's form

$\therefore$ the solution is

$$z=cx-c^2$$

or $\qquad xy=cx-c^2$

Q. 14. Solve the differential equation

$$y\left(\frac{dy}{dx}\right)^2+x^3\frac{dy}{dx}-x^2y=0.$$

Sol. Putting $\frac{dy}{dx}=p,$ we have

$$yp^2+x^3p-x^2y=0$$

Multiplying throughout by y, we have

$$y^2p^2+x^3py-x^2y^2=0 \qquad \text{....(i)}$$

Put $\quad x^2=X$ and $y^2=Y$

$\therefore \quad 2x\,dx=dX$ and $2y\,dy=dY$

$$\therefore \quad \frac{2y\,dy}{2x\,dx}=\frac{dY}{dX}=P,\ say.$$

$$\Rightarrow \quad \frac{y}{x}\,\frac{dy}{dx}=P$$

$$\Rightarrow \quad \frac{y}{x}\,p=P$$

$$\Rightarrow \quad py=Px.$$

Substituting in equation (i), we have

$$(Px)^2+x^3.Px-x^2y^2=0.$$

$$\Rightarrow \quad P^2x^2+x^4P-x^2y^2=0$$

$$\Rightarrow \quad P^2+x^2P-y^2=0$$

$\Rightarrow \quad P^2+XP-Y=0 \qquad \Theta \qquad x^2=X \text{ and } y^2=Y$

$\Rightarrow \quad Y=PX+P^2$

which is a Clairaut's form

$\therefore$ the solution is

$$Y=CX+C^3$$

$\Rightarrow \quad y^2=cx^2+c^2$

Q. 15. Solve the differential equation

$$y^2(y-xp)=x^4p^2$$

Sol. The = n can be written as

$$y\text{-}xp=\frac{x^4p^2}{y^2}$$

or $$\frac{y-xp}{y^2}=\frac{x^4p^2}{y^4}$$

or $$\frac{1}{y}-\frac{xp}{y^2}=\frac{x^4p^2}{y^4} \qquad \text{......(i)}$$

put $x=\dfrac{1}{X}$ and $y=\dfrac{1}{Y}$

$\therefore \quad dx=-\dfrac{1}{X^2}\,dX$ and $dy=-\dfrac{1}{Y^2}\,dY$

$$\therefore \quad p=\frac{dy}{dx}=\frac{-\frac{1}{Y^2}dy}{-\frac{1}{X^2}dX}=\frac{X^2}{Y^2}\frac{dY}{dX}=\frac{y^2}{x^2}P$$

$\therefore \qquad \frac{x^2 p}{y^2} = P$

Substituting in = n (i) we have

$$\frac{1}{y} - \frac{px}{y^2} = P^2$$

$\Rightarrow \qquad \frac{x}{y} - \frac{x^2 p}{y^2} = p^2$

$\Rightarrow \qquad -\frac{x}{y} - P = P^2$

$\Rightarrow \qquad \frac{Y}{X} - P = P^2, as \;\; x = \frac{1}{X} \;\; and \;\; y = \frac{1}{Y}$

$\Rightarrow \qquad Y - PX = P^2X$

$\Rightarrow \qquad Y = PX + P^2X$

which is a Clairaut's form.

$\therefore$ the Solution is

$$Y = CX + C^2X$$

$\Rightarrow \qquad \frac{1}{y} = \frac{c}{x} + \frac{c^2}{x}$

$\Rightarrow \qquad x = y(c + c^2)$

$\Rightarrow \qquad x = cy(1 + c).$

Unit-II	**Lesson No. 7**
B.A. 2nd Semester	**Subject : Mathematics**

Equations of the second order :

The typical form of equations of the second order is

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + \text{Qy=X} \qquad ....(1)$$

In the symbolic form it is written as

$$[D^2+PD+Q] \ \ y = X$$

where P,Q are constants, X is a function of *x or a constant and*

$$D=\frac{d}{dx}, \quad D^2=\frac{d^2}{dx^2}$$

we shall deal with two forms of this equation.

(i) when the right hand side member is zero,

(ii) when the right hand side member is a function of *x*.

Let us consider the first form

i.e. Equation with right hand member zero

Let the equation be

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + \text{Qy=0}$$

As a trial solution of (2) Let us take $y=e^{mx}$. Then if we put $y=e^{mx}$ in the left hand side of (2) it must satisfy the equation i.e., we have

$$(m^2+Pm+Q)\, e^{mx}=0$$

Since $e^{mx} \neq 0 \therefore m^2+Pm+Q=0$ ....(3)

The equation (3) is called the auxiliary equation of (2).

Let m_1, m_2 be the roots of equation (3)

Then $y= e^{m_1 x}$ *and* $y = c^{m_2 x}$ are abviously solutions of (2). Also applying direct substitution we see that

$$y=c_1 e^{m_1 x} \text{ and } y=c_2 e^{m_2 x}$$

satisfy that equation (2) and as such are solutions of (2)

We shall now consider the nature of the general solution of the equation (2) according as the roots of the Auxilary equation (3) are

(i) real and distinct

(ii) real and equal

(iii) Imaginary

(i) Auxilary equation having real and distinct roots ;

If m_1, m_2 are real and distinct, then

$$y=c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

is the general solution. Since it satisfies the equation and contains two independent arbitrary constants equal in number to the order of the equation.

(ii) Auxilary equation having two real and equal roots:

If the auxilary equation has two equal roots, then the preceding paragraph does not lead to the general solution.

For if $m_1=m_2=\alpha$, say, then the solution of the preceding paragraph assumes the form

$$y = (c_1 + c_2) e^{\alpha x} = c e^{\alpha x} \text{ where } c = c_1 + c_2$$

which is not the general solution, because the equation is of the second order and it involves only one constant.

$\therefore$ in this case the following method is adopted for finding the general solution. Since the auxilary equation (3) has two equal roots each being equal to α, it follows that the differential equation (2) assumes the form

$$\frac{d^2y}{dx^2} + 2\alpha \frac{dy}{dx} + \alpha^2 y = 0$$

Let y= $y = e^{\alpha x}$ v, where v is a function of x be a trial solution of the equation.

Substituting this value of y in the left side of above equation.

we have

$$e^{\alpha x} \frac{d^2v}{dx^2} = 0 \text{ i.e. } \frac{d^2v}{dx^2} = 0, \text{ Since } e^{\alpha x} \neq o$$

Now integrating twice, we have

$$\frac{dv}{dx} = c_2$$

and $\therefore$ $v = c_1 + c_2 x$

Hence the solution of (2) in this case is

$$y = (c_1 + c_2 x) \; e^{\alpha x}$$

Since it satisfies (2) and contains two arbitrary constants.

(iii) Auxilary equation having a pair of complex roots :

Let $m_1 = a + ib$, $\therefore$ $m_2 = a - ib$

$\therefore$ the general solution of (2) is

$$y = c_{1e}{}^{(a+ib)x} + c_2 \, e^{(a-ib)}x$$

$$= e^{ax}(c_1 e)^{ibx} + c_2\, e^{-ibx}$$

$$= e^{ax}[c_1(\cos bx + i\sin x) + c_2(\cos bx - i\sin bx)]$$

$$= e^{ax}[(c_1+c_2)\cos bx + i(c_1-c_2)\sin bx]$$

$$= e^{ax}[A\cos bx + B\sin bx]$$

where $A=c_1+c_2$ and $B=i(c_1-c_2)$

are the arbitrary constants which may be given any real value we like.

Note I. In case the auxilary equation has two equal roots each equal to m, then the solution is

$$y=(c_1+c_2x)\, e^{mx}$$

If there are three equal roots, each equal to m, then the solution is

$$y=(c_1+c_2x+c_3x^2)\, e^{mx}$$

Note II. The general solutions is called the Complementary Function (C.F.)

Q. 1. Solve the differential equation

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}-10y=0$$

Sol. The equation can be written as

$(D^2+3D-10)\ y=0$

The auxilary equation is

$m^2+3m-10=0$

$\Rightarrow$ $(m+5)(m-2)=0$

$\Rightarrow$ $m=-5,2$

$\therefore$ the general solution is

$y = c_1 e^{-5x} + c_2 e^{2x}$

Q. 2. Solve $(D^3 - D^2 - 2D - 12)y = 0$

Sol. The auxilary equation is

$m^3 - m^2 - 2m - 12 = 0$

$\Rightarrow$ $(m-3)(m^2+2m+4)=0$

Now m=3 satisfies the equation

$\therefore$ applying synthetic division, we have

3 |	1	–1	–2	–12
		3	6	12
	1	2	4	– 0

$\therefore$ the remaining equation is

$m^2 + 2m + 4 = 0$

$\therefore$ m=3

and m is given by

$m^2+2m+4=0$

$$\therefore \quad m = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm 2\sqrt{3}i}{2}$$

$$= -1 \pm \sqrt{3}\; i$$

$\therefore$ the solution of differential equation is

$$y = c_1 e^{3x} + c_2 e^{(-1+\sqrt{3}\ i)}x \quad + c_2 e^{(-1-\sqrt{3}\ i)x}$$

$$= c_1 e^{3x} + e^{-x}(A\ \cos\sqrt{3}\,x + B\sin\sqrt{3}\,x)$$

Q. 3. Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2}$+4y=0

Sol. The equation can be written as

$(D^3–3D^2+4)$ y=o

The auxilary equation is

$m^2–3m^2+4=0$

m=– 1 satisfies the *equation*

∴ by synthetic discussion, we have

–1	1	–3	0	4
		–1	4	–4
	1	–4	4	0

∴ the remaining equation is

$m^2–4m+4=0$

∴ $(m–2)^2=0$

⇒ m=2,2

Hence m=–1,2,2

∴ the general solution is

$$y = c_1 e^{-x} + (c_2 + c_3 x)e^{2x}$$

Q. 4. Solve the differential equation

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=0$$

Sol. The equation can be written as

$(D^2+6D+13)y=0$

The auxilary equation is

$m^2+6m+13=0$

$\therefore \quad m=\dfrac{-6\pm\sqrt{36-52}}{2}$

$=\dfrac{-6\pm\sqrt{-16}}{2}$

$=\dfrac{-6\pm 4\,i}{2}=3\pm 2i$

$\therefore$ the general solution is

$$y=c_1e^{-3x}\,(A\cos 2x+B\sin 2x)$$

Q. 5. Solve the equation

$(D^3+6D^2+11D+6)\ y=0$

Sol. The auxilary equation is

$m^3+6m^2+11m+6=0$

$m=-1$ satisfies the equation

$\therefore$ By synthetic division, we have

−1	1	6	11	6
		−1	−5	−6
	1	5	6	0

The remaining equation is

$m^2+5m+6=0$

$\Rightarrow \quad (m+2)(m+3)=0$

$\Rightarrow \quad m=-2, -3$

$\therefore$ the roots are $-1, -2, -3$

$\therefore$ the general solution is

$$y=c_1e^{-x}+c_2e^{-2x}+c_3e^{-3x}$$

Q. 6. Solve the equation

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+4y=0$$

Sol. The equation can be written as

$(D^2+5D+4)\ y=0$

The auxilary equation is

$m^2+5m+4=0$

$\Rightarrow \quad (m+1)(m+4)=0$

$\therefore \quad m=-1, -4$

Hence the solution is

$$y=c_1e^{-x}+c_2e^{-4x}$$

Q. 7. Solve the equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Sol. The equation can be written as

$$(D^2 - 2D + 1)\, y = 0$$

The auxilary = n is

$$m^2 - 2m + 1 = 0$$

$\Rightarrow$ $(m-1)^2 = 0$

$\Rightarrow$ $m = 1, 1$

$\therefore$ the solution is

$$y = (c_1 + c_2 x)e^{x}$$

Q.8. Solve the equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 0$$

Sol. The equation can be written as

$$(D^2 + 6D) = 0$$

The auxilary equation is

$$m^2 + 6m = 0$$

$\Rightarrow$ $m(m+6) = 0$

$\Rightarrow$ $m = 0, -6$

$\therefore$ the solution is

$$y = c_1 e^{-6x} + c_2 e^{0x} = c_1 e^{-6x} + c_2$$

Q. 9. Solve the equation

$$\frac{d^2y}{dx^2}+(a+b)\frac{dy}{dx}+aby=0$$

Sol. The equation can be written as

$[D^2+(a+b)D+ab]y=0$

The auxilary equation is

$m^2+(a+b)m+ab=0$

or $(m+a)(m+b)=0$

$\Rightarrow$ $m=-a, -b$

$\therefore$ the solution is

$$y=c_1e^{-ax}+c_2e^{-bx}$$

Q.10. Solve the equation

$[D^2+6D+25]\ y=0$

Sol. The auxilary is

$m^2+6m+25=0$

$$\therefore \quad m=\frac{-6\underline{+}\sqrt{36-100}}{2}$$

$$=\frac{-6\underline{+}\sqrt{-64}}{2}-3\underline{+}4\,i$$

$\therefore$ the general solution is

$$y=c_1e^{(-3+4\,i)x}+c_2e^{(-3-4\,i)x}$$

$$=e^{-3x}[c_1e^{4_i x}+c_2e^{-4_i x}]$$

$$= e^{-3x}[c_1(\cos 4x + i \sin 4x) + c_2(\cos 4x - i \sin 4x)]$$

$$= e^{-3x}[(c_1 + c_2)\cos 4x + i\,(c_1 - c_2)\sin 4x]$$

$\therefore$ $\quad y = e^{-3x}\,(A \cos 4x + B \sin 4x)$

where $A = c_1 + c_2$ and $B = i\,(c_1 - c_2)$

Q.11. Solve the equation

$(D^3+D^2-D-1)\; y = 0$

Sol. The auxilary equation is

$m^3 + m^2 - m - 1 = 0$

m = 1 satisfies the equation

$\therefore$ By synthetic discussion, we have

1 |	1	1	–1	–1
		1	2	1
	1	2	1	0

The remaining equation is

$m^2+2m+1 = 0$

$\Rightarrow$ $(m+1)^2=0$

$\Rightarrow$ $m = -1, -1$

$\therefore$ the roots are 1, –1,–1

$\therefore$ the solution is

$$y = (c_1+c_2x)\; e^{-x} + c_3e^x$$

Q. 12. Solve the equation

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$$

Sol. The equation can be written as

$(D^2 - 7D + 12)\ y = 0$

The auxilary equation is

$m^2 - 7m + 12 = 0$

$\Rightarrow \quad (m-3)(m-4) = 0$

$\Rightarrow \quad m = 3\ ,\ 4$

$\therefore$ the solution is

$$y = c_1 e^{3x} + c_2 e^{4x}$$

Q.13. Solve the equation

$$2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$$

Sol. The equation can be written as

$(2D^2 - 3D + 1)\ y = 0$

The auxilary equation is

$2m^2 - 3m + 1 = 0$

$$\therefore \quad m = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4} = 1, \frac{1}{2}$$

$\therefore$ the solution is

$$y = c_1 e^{x} + c_2 e^{x/2}$$

Q. 14. Solve the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Sol. The = n can be written as

$(D^2-4D+4)\ y = 0$

The auxilary = n is

$m^2 - 4m + 4 = 0$

$\Rightarrow \quad (m-2)^2 = 0$

$\Rightarrow \quad m = 2\ ,\ 2$

$\therefore$ the solution is

$y = (c_1 + c_2 x)\ e^{2x}$

Q. 15. Solve the differential equation

$[D^2 - 2mD + (m^2+n^2)]\ y = 0$

Sol. The auxilary equation is

$D^2 - 2mD + (m^2+n^2) = 0$

$\therefore \quad D = 2m \pm \sqrt{4m^2 - 4m^2 - 4n^2}$

$$= \frac{2m \pm \sqrt{-4n^2}}{2}$$

$$= \frac{2m \pm 2ni}{2} = m \pm in$$

$\therefore$ the general solution is

$$y = e^{mx}\,(A\cos nx + B\sin nx)$$

Q. 16. Solve the equation

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$$

Sol. The equation can be written as

$(D^3-4D^2+5D-2)\ y = 0$

The auxilary equaton is

$m^3-4m^2+5m-2=0$

$m=1$ satisfies the *equation*

∴ by Sythetic division, we have

1	1	–4	5	–2
		1	–3	2
	1	–3	2	0

The remaining equation is

$m^2-3m+2=0$

⇒ $(m-1)(m-2)=0$

⇒ $m=1,\ 2$

∴ the roots are 1, 1, 2

∴ the solution is

$y=(c_1+c_2x)e^x+c_3e^{2x}$

Q. 17. Solve the equation

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$$

Sol. The equation can be written as

$(D^3 - 2D^2 + 4D - 8)\ y = 0$

The auxilary equation is

$m^2 - 2m^2 + 4m - 8 = 0$

m=2 satisfies the equation

∴ the Synthetic division, we have

2	1	–2	4	–8
		2	0	8
	1	0	4	0

The remaining equation is

$m^2 + 4 = 0$

$\Rightarrow \quad m^2 = -4$

$\Rightarrow \quad m = \pm 2i$

∴ the roots are 2, $\pm 2i$

Hence the solution is

$$y = c_1 e^{2x} + A\cos 2x + B\sin 2x$$

Q. 18. Solve the equation

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0, \text{ given that when } t=0, x=0$$

and $\frac{dx}{dt} = 0$

Sol. The equation can be written as

$(D^2 - 3D + 2)\, x = 0$

The auxilary equation is

$m^2 - 3m + 2 = 0$

$\Rightarrow \quad (m-1)(m-2) = 0$

$\Rightarrow \quad m = 1, 2$

$\therefore$ the solution is

$$x = c_1 e^t + c_2 e^{2t}$$

when $t = 0, x = 0$

$\Rightarrow \quad 0 = c_1 + c_2$ .....(1)

Again $\frac{dx}{dt} = c_1 e^t + 2c_2 e^{2t}$

when $t = 0, \frac{dx}{dt} = 0$

$\Rightarrow \quad 0 = c_1 + 2c_2$ ....*(ii)*

Subtracting (ii) from (i), we have

$-c_2 = 0 \quad \Rightarrow \quad c_2 = 0$

$\therefore \quad 0 = c_1 + 0$

$\Rightarrow \quad c_1 = 0$

$\therefore$ the solution is

$x = 0$

Q.19. Show that the solution of

$x + K\,x + \mu x = 0$ *is*

$x = e^{-\frac{1}{2}Kt}$ *(A cos nt + B sin nt)*

where $n^2 = \mu - \frac{1}{4}K^2$ and n is real, dots

denoting differentiation w.r.t. t

Sol. The equation can be written as

$$\frac{d^2x}{dt} + K\frac{dx}{dt} + \mu x = 0$$

or $(D^2+KD+\mu)\ x=0$

The auxilary equation is

$m^2+Km+\mu=0$

$\therefore$ $$m = \frac{-K \pm 2\sqrt{K^2 - 4\mu}}{2}$$

$$= \frac{-K \pm 2\sqrt{K^2/4 - \mu}}{2}$$

$$= \frac{-K \pm 2\sqrt{-n^2}}{2}$$

$$= \frac{-K \pm 2ni}{2} = \frac{-K}{2} \pm ni$$

$\therefore$ the solution is

$$x = e^{-\frac{1}{2}Kt}(\text{A cos nt + B sin nt})$$

Equations with right hand member a function of x :

Let the equation be

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + \text{Qy=x (function of x)} \qquad .....(1)$$

If y= ϕ (x) be the general solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + \text{Qy=0} \qquad ....(2)$$

and y= ψ (x) be any particular solution of (1), then
y= ϕ (x) + ψ (x) is the the general solution of (1)

The result can be established by direct substitution.

Thus, substituting $y = \phi(x) + \psi(x)$ in the left hand side of (1), we have

$$\left|\frac{d^2\phi}{dx^2} + P\frac{d\phi}{dx} + Q\phi\right| + \frac{d^2\Psi}{dx^2} + P\frac{d\Psi}{dx} + Q\Psi\Bigg|$$

The first group of terms is zero, since y= ϕ(x) is solution of (2) and the second group of terms is equal to X since y=ψ(x) is a solution of (1). Hence y= ϕ(x)+ψ(x) is a solution of (1) and it is the general solution, since the number of independent arbitrary constants in it is two, ϕ(x) being the general solution of (2)

We have two solutions

(1) The Complementary Function (C.F.) when the equation is equated to zero. This has already been discussed in Lesson No. 6.

(2) The particular integral (P.I.) when the equation is equated to X.

Art. 1. The Inverse operator $\frac{1}{D-A}$:–

We have to find the value of $\frac{1}{D-a}$ u, where u is a function of x.

Let $\frac{1}{D-a} u = z$

Operating upon both sides by D-a, we get

$(D-a)\, z = (D-a)\,(D-a)^{-1} u$

$\Rightarrow (D-a)\, z = u$

or $\frac{dz}{dx} - az = u$

which is linear differential equation of the first order

$\therefore$ Its solution is

$z = e^{ax} \int u e^{-ax}\, dx$, The I.F. $= e^{\int -a dx} = e^{-ax}$

Omitting the constant of integration

Thus $\frac{1}{D-a} u = e^{ax} \int u e^{-ax}\, dx$ ....(1)

Cor. $\frac{1}{D-a} e^{mn} = \frac{1}{m-a} e^{mn}$, $m \neq a$

Important Forms of Particular Integrals :

(1) Value of $\frac{1}{f(D)} e^{ax}$

$D(e^{ax}) = ae^{ax}$

$D^2(e^{ax}) = a^2 e^{ax}$

$D^3(e^{ax}) = a^3 e^{ax}$

....

$D^n (ea^x) = a^n e^a x$

Therefore $f(D)\, e^{ax} = f(a)\, e^{ax}$

Operating by $\frac{1}{f(D)}$ on the both sides of this equation,

We have

$$e^{ax} = f(a) \left\{ \frac{1}{f(D)} e^{ax} \right\}$$

$$\therefore \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ provided } f(a) \neq 0$$

Case of failure

If f(a)=0, then D–a is a factor of f(D), this is a case of failure

Suppose f(D)=(D–a) F(D) and F(D) does not contain the factor D–a, then

$$\frac{1}{f(D)} e^{ax} = \frac{1}{D-a} \cdot \frac{1}{F(D)} e^{ax}$$

$$=\frac{1}{F(a)}\frac{1}{D-a}e^{ax}$$

$$=\frac{1}{F(a)}e^{ax}\int e^{-ax}e^{ax}{}_{dx}$$

$$\left[\because \frac{1}{D-a}=e^{ax}\int e^{-ax}e^{ax}dx\right]$$

$$=\frac{1}{F(a)}e^{ax}.x$$

$$\therefore\ \frac{1}{f(D)}e^{ax}=\frac{1}{D-a}\frac{1}{F(D)}e^{ax}=\frac{xe^{ax}}{F(a)}$$

Similarly if $(D–a)^2$ occurs as a factor in f(D), say, then $f(D)=(D–a)^2 F(D)$

$$\therefore\ \frac{1}{f(D)}e^{ax}=\frac{1}{(D-a)^2}\frac{1}{F(D)}e^{ax}$$

$$=\frac{1}{F(a)}\frac{1}{D-a}\frac{1}{D-a}e^{ax}$$

$$=\frac{1}{F(a)}\frac{1}{D-a}xe^{ax}$$

$$=\frac{1}{F(a)}e^{ax}\int e^{-ax}.xe^{ax}dx$$

$$=\frac{1}{F(a)}e^{ax}\int xdx$$

$$= \frac{1}{F(a)} e^{ax} \frac{x^2}{2!}$$

$$= \frac{x^2}{2!} \quad \frac{e^{ax}}{F(a)}$$

Similarly $\frac{1}{(D-a)^3} e^{ax} = \frac{1}{D-a} \frac{1}{(D-a)^2} e^{ax}$

$$= \frac{1}{D-a} \left(\frac{x^2}{2!} e^{ax} \right)$$

$$= e^{ax} \int \frac{x^2}{2!} dx \, x$$

$$= \frac{x^3}{3!} \quad e^{ax}$$

Continuing like this we get after n operations.

$$\frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$$

Thus we have the results as

1. $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$, provided f(a)≠0

2. $\frac{1}{D+a} e^{ax} = \frac{1}{a+a} e^{ax} = \frac{1}{2a} e^{ax}$, as is substituted for D.

3. $\frac{1}{D-a} e^{ax} = x\, e^{ax}$

4. $\frac{1}{(D-a)^2} e^{ax} = \frac{x^2}{2!} e^{ax}$

5. $\frac{1}{(\mathrm{D}-\mathrm{a})^3} e^{ax} = \frac{x^3}{3!} e^{ax}$

....

6. $\frac{1}{(\mathrm{D}-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$

Q.1. Solve the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{-x}$$

Sol. The equation can be written as

$(D^2+D+1)\, y = e^{-x}$

The auxilary equation is

$m^2+m+1=0$

$$\therefore \ m = \frac{-1 \pm \sqrt{1-14}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\therefore \text{ C.F.}=e^{\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.}=\frac{1}{D^2+D+1}e^{-x}$$

$$=\frac{1}{\left(-1^2\right)+\left(-1\right)+1}e^{-x}=\frac{1}{1-1+1}e^{-x}=e^{-x}$$

Hence the complete solution is

$$\text{y}=e^{-\frac{1}{2}x}\left(A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x\right)+e^{-x}$$

Q.2. Solve the equation

$(D^2+4D+3)y=e^{-3x}$

Sol. The auxilary equation is

$m^2+4m+3=0$

$\Rightarrow (m+1)(m+3)=0$

$\Rightarrow m=-1,-3$

$\text{C.F.}=c_1e^{-x}+c_2e^{-3x}$

$$\therefore \text{ P.I.}=\frac{1}{D^2+4D+3}e^{-3x}$$

$$=\frac{1}{(D+1)(D+3)}e^{-3x}\text{, Here } f(\text{D})=0$$

$$=\frac{1}{(D+3)(-3+1)}e^{-3x}$$

$$=-\frac{1}{2}\frac{1}{(D+3)}e^{-3x}$$

$$=-\tfrac{1}{2}xe^{-3x}$$

$\therefore$ the complete solution is

$$y=c_1e^{-x}+c_2e^{-3x}-\tfrac{1}{2}xe^{-3x}$$

Q.3. Solve the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^x$$

Sol. The equation can be written as

$(D^2-3D+2)\ y=e^x$

The auxilary equation is

$m^2-3m+2=0$

$\Rightarrow (m-1)\ (m-2)=0$

$\Rightarrow m=1,\ 2$

$\therefore$ C.F.$=c_1e^x+c_2e^{2x}$

$$\text{P.I.}=\frac{1}{D^2-3D+2}e^x$$

$$=\frac{1}{(D-2)(D-1)}e^x$$

$$=\frac{1}{(1-2)(D-1)}e^x$$

$$=-\frac{1}{D-1}e^x$$

$=-x\, e^x$

$\therefore$ the general solution is

$y=c_1e^x+c_2e^{2x}-xe^x$

Q.4. Solve completely the equation

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=5e^{2x}$$

Sol. The equation can be written as

$(D^2-4D+3)y=5e^{2x}$

The auxilary equation is

$m^2-4m+3=0$

or $(m-1)\,(m-3)=0$

$m=1,\ 3$

C.F. $=c_1e^x+c_2e^{3x}$

Again P.I.$=\frac{1}{D^2-4D+3}5e^{2x}$

$$=5\frac{1}{D^2-4D+3}e^{2x}$$

$=5\frac{1}{2^2-4\times 2+3}e^{2x}$

$=5\frac{1}{4-8+3}e^{2x}$

$=-5e^{2x}$

$\therefore$ the complete solution is

y=$c_1e^x+c_2e^{3x}-5e^{2x}$

Q.5. Solve $(D^2-4D+3)\ y=e^{3x}$

Sol. As in Q 4

C.F.=$c_1e^x+c_2e^{3x}$

P.I.=$\frac{1}{D^2-D+3}e^{3x}$

$=\frac{1}{(D-3)(D-2)}e^{3x}$

$=\frac{1}{D-3}\cdot\frac{1}{3-1}e^{3x}$

$=\frac{1}{2}\frac{1}{D-3}e^{3x}$

$=\frac{1}{2}xe^{3x}$

$\therefore$ the complete solution is

$y=c_1e^x+c_2e^{3x}+\frac{1}{2}xe^{3x}$

Q.6. Solve the equation

$(D-a)^3\, y=e^{ax}$

Sol. The auxilary equation is

$(m-a)^3=0$

$\Rightarrow$ $m=a,\ a,\ a$

$\therefore$ C.F.$=(c_1+c_2x+c_3x^2)e^{ax}$

$$\text{P.I.}=\frac{1}{(D-a)^3}\, e^{ax}$$

$$=\frac{x^3}{3!}e^{ax}=\frac{1}{6}x^3e^{ax}$$

$\therefore$ the complete solution is

$$y=(c_1+c_2x+c_3x^2)e^{ax}+\frac{1}{6}x^3e^{ax}$$

Q.7. Solve the equation :

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^x$$

Sol. The equation can be written as

$(D^2+3D+2)\, y=e^x$

The auxilary equation is

$m^2+3m+2=0$

$$\therefore\quad m=\frac{-3\pm\sqrt{9-8}}{2}=\frac{-3\pm 1}{2}=-1,\ -2$$

$\therefore$ C.F.$=c_1e^{-x}+c_2\ e^{-2x}$

$$\text{P.I.}=\frac{1}{D^2+3D+2}e^x$$

$$=\frac{1}{1^2+3\cdot 1+2}e^x$$

$$=\frac{1}{1+3+2}e^x=\frac{1}{6}e^x$$

$\therefore$ the solution (complete) is

$$y=c_1e^{-x}+c_2\ e^{-2x}+\frac{1}{6}e^x$$

Q.8. Solve the equation

$$\frac{d^2y}{dx^2}+2p\frac{dy}{dx}+(p^2+q^2)y=e^{ax}$$

Sol. The equation can be written as

$[D^2+2pD+(p^2+q^2)]y=0$

The auxiliary equation is

$m^2+2pm+(p^2+q^2)=0$

$$\therefore \quad m=\frac{-2p\pm\sqrt{4p^2-4p^2-4q^2}}{2}$$

$$=\frac{-2p\pm\sqrt{-4q^2}}{2}$$

$$=\frac{-2p\pm 2qi}{2}$$

$$=-p\pm qi$$

∴ C.F.$=e^{-px}$(A cos qx+B sin qx)

$$\text{P.I.}=\frac{1}{D^2+2\text{pD}+\text{p}^2+\text{q}^2}\,\text{e}^{\text{ax}}$$

$$=\frac{1}{a^2+pa+p^2+q^2}\,e^{ax}$$

$$=\frac{1}{(a+p)^2+q^2}\,e^{ax}$$

∴ the complete solution is

$$y=e^{-px}\,(\text{A cos } qx+\text{B sin } qx)+\frac{1}{(a+p)^2+q^2}\,e^{ax}$$

Q.9. Solve the equation :

$$\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=e^x$$

The equation can be written as

$(D^3-3D^2+4D-2)\ y=e^x$

The auxilary equation is

$m^3-3m+4m-2=0$

m=1 satisfies the equation.

∴ By synthetic division, we have

1	1	–3	4	–2
		1	–2	2
	1	–2	2	0

The remaining equation is

$m^2 - 2m + 2 = 0$

$$\therefore \quad m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2} = 1 \pm i$$

∴ m=1, $1 \pm i$

∴ $C.F. = c_1 e^x + e^x (A \cos x + B \sin x)$

$$P.I. = \frac{1}{D^3 - 3D^3 + 4D - 2} e^x$$

$$= \frac{1}{(D-1)(D^2 - 2D + 2)} e^x$$

$$= \frac{1}{(D-1)(1^2 - 2 \times 1 + 2)} e^x$$

$$= \frac{1}{(D-1)} \frac{1}{1-2+2} e^x$$

$$= \frac{1}{D-1} e^x$$

$=xe^{x.}$

$\therefore$ the complete solution is

$y=(c_1+\text{A}\cos x+\text{B}\sin x)\, e^x+xe^x$

$=(c_1+\text{A}\cos x+\text{B}\sin x+x)\, e^{x.}$

Q.10. Solve the equation

$$\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+3\frac{dy}{dx}+\text{y}=\text{e}^{-\text{x}}.$$

Sol. The equation can be written as

$(\text{D}^3+3\text{D}^2+3\text{D}+1)y=e^{-x}$

The auxilary equation is

$m^3+3m^2+3m+1=0$

$\Rightarrow$ $(m+1)^3=0$

$\Rightarrow$ $\text{m}=-1, -1, -1$

$\therefore$ C.F.$=(c_1+c_2x+c_3x^2)e^{-x}$

$$\text{P.I.}=\frac{1}{D^3+3D^2+3D+1}e^{-x}$$

$$=\frac{1}{(D+1)^3}e^{-x}$$

$$=\frac{x^3}{\underline{|3}}\text{e}^{-\text{x}}=\frac{x^3}{6}\text{e}^{-\text{x}}$$

$\therefore$ the complete solution is

$$y=(c_1+c_2n+c_3x^2)\ e^{-x}+\frac{x}{6}e^{-x}$$

$$=\left(c_1+c_2x+c_3x^2+\frac{1}{6}x^3\right)e^{-x}$$

Q.11. Solve the equation :

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=e^{2x}$$

the equation can be written as

$(D^2+5D+6)\ y=e^{2x}$

The auxilary equationis

$m^2+5m+6=0$

$\Rightarrow$ $(m+2)\ (m+3)=0$

$\Rightarrow$ $m=-2, -3$

$\therefore$ C.F.$=c_1\ e^{-2x}+c_2e^{3x}$

$$\text{P.I.}=\frac{1}{D^2+5D+6}e^{2x}$$

$$=\frac{1}{2^2+5\times2+6}e^{2x}$$

$$=\frac{1}{4+10+6}e^{2x}$$

$$=\frac{1}{20}e^{2x}$$

$\therefore$ the complete solution is

$$y=c_1e^{-2x}+c_2\ e^{-3x}+\frac{1}{20}e^{2x}$$

Q.12. Solve the equation :

$$4\frac{d^2y}{dx^2}+4\frac{dy}{dx}-3y=e^{2x}$$

Sol. The equation can be written as

$(4D^2+4D-3)\ y=e^{2x}$

The auxilary equation is

$4m^2+4m-3=0.$

$$\therefore \quad m=\frac{-4\pm\sqrt{16+48}}{8}=\frac{-4\pm\sqrt{64}}{8}$$

$$=\frac{-4\pm 8}{8}=\frac{-3}{2},\frac{1}{2}$$

$\therefore$ C.F$=c_1\ e^{1/2x}+c_2\ e^{-3/2x}$

$$\text{P.I.}=\frac{1}{4D^2+4D-3}e^{2x}$$

$$=\frac{1}{4\times 2^2+4\times 2\text{-}}e^{2x}$$

$$=\frac{1}{16+8-3}e^{2x}$$

$=\frac{1}{21}e^{2x}.$

the complete solution is

$$y=c_1\ e^{1/2x}+c_2\ e^{-3/2x}+\frac{1}{21}\ e^{2x}$$

Q.13. Solve the equation :

$(D+3)^2y=25e^{2x}$

Sol. The auxilary equation is

$(m+3)^2=0$

$\Rightarrow$ $m=-3, -3$

$\therefore$ C.F. $=(c_1+c_2x)\ e^{-3x}$

$$P.I.=\frac{1}{(D+3)^2}25\ e^{2x}$$

$$=25\frac{1}{(2+3)^2}e^{2x}$$

$$=25.\frac{1}{25}e^{2x}=e^{2x}$$

$\therefore$ The complete solution is

$y=(c_1+c_2x)\ e^{-3x}+e^{2x}$

Q.14. Solve the equation

$(D^2+9)\ y=9e^{3x}$

Sol. The auxilary equation is

$m^2+9=0$

$\Rightarrow$ $m^2=-9$

$\Rightarrow$ $m=\pm 3i$

$\therefore$ C.F.=A cos 3x+B sin 3x

$$\text{P.I.}=\frac{1}{D^2+9} \cdot 9\, e^{3x}$$

$$=9\frac{1}{3^2+9}e^{3x}$$

$$=\frac{9}{18}e^{3x}=\frac{1}{2}e^{3x}$$

$\therefore$ the complete solution is

$$y=A\cos 3x+B\sin 3x+\tfrac{1}{2}e^{3x}$$

Q.15. Solve the equation

$(D^2+6D+9)\ y=2e^{-3x}$

Sol. The auxilary equation is

$m^2+6m+9=0$

$\Rightarrow$ $(m+3)^2=0$

$\Rightarrow$ $m=-3, -3$

C.F.$=(c_1+c_2x)\ e^{-3x}$

$$\text{P.I.}=\frac{1}{D^2+6D+9}2e^{-3x}$$

$$=2\frac{1}{(D+3)^2}\ e^{-3x}$$

$$=2.\frac{x^2}{2!}e^{-3x}$$

$= x^2e^{-3x}$

$\therefore$ the complete solution is

$y=(c_1+c_2x)\ e^{-3x}+x^2e^{-3x}$

$\therefore$ $y=(c_1+c_2x+x^2)\ e^{-3x}$

Q.16. Solve the equation

$$\frac{d^2y}{dx^2}-a^2y=e^{ax}$$

Sol. The equation can be written as

$(D^2-a^2)\ y=e^{ax}$

The auxilary equation is

$m^2-a^2=0$

$\Rightarrow$ $m^2=a^2$

$\Rightarrow$ $m=\pm a$

$\therefore$ C.F.$=c_1e^{ax}+c_2e^{-ax}$

$$\text{P.I.}=\frac{1}{D^2-a^2}\ e^{ax}$$

$$=\frac{1}{(D+a)(D-a)}\ e^{ax}$$

$$=\frac{1}{(a+a)(D-a)}e^{ax}$$

$$=\frac{1}{2a}xe^{ax}=\frac{x}{2a}e^{ax}$$

$\therefore$ the complete solution is

$$y=c_1e^{ax}+c_2e^{-ax}+\frac{x}{2a}e^{ax}$$

Q.17. Solve the differential equation

$$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-\frac{dy}{dx}+y=e^x$$

Sol. The equation can be written as

$(D^3-D^2-D+1)\ y=e^x$

The auxilary =n is

$m^3-m^2-m+1=0$

m=1 satisfies the equation

$\therefore$ By synthetic division, we have

1	1	–1	–1	1
		1	0	–1
	1	0	–1	0

The remaining equation is

$m^2-1=0$

$\Rightarrow$ $m^2=1$

$\Rightarrow$ $m=\pm 1$

$\therefore$ The roots are 1, 1, –1

$\therefore$ C.F.$=(c_1+c_2x)\ e^x+c_3e^{-x}$

$$\text{P.I.}=\frac{1}{D^3-D^2+D+1}\ e^x$$

$$=\frac{1}{(D-1)^2(D+1)}e^x$$

$$=\frac{1}{(D-1)^2(1+1)}\ e^x$$

$$=\frac{1}{2}.\frac{x^2}{2!}e^x=\frac{1}{4}x^2\ e^x.$$

Complete Solution is

y = C.F. + P.I.

$$=(c_1+c_2x)\ e^x+\ c_3e^{-x}+\frac{1}{4}x^2e^x$$

Q.18. Solve the equation

$(D^3–3D^2+4)\ y=e^{3x}$

Sol. The auxilary equation is

$m^3–3m^2+4=0$

m=–1 satisfies the equation

$\therefore$ By Synthetic division we have

–1	1	–3	0	4
		–1	4	–4
	1	–4	4	0

The remaining equation is

$m^2–4m+4=0$

= $(m–2)^2=0$

$\Rightarrow$ $m=2, 2 \Rightarrow m= –1, 2, 2$

$\therefore$ $C.F.=c_1e^{-x}+(c_2+c_3x)\ e^{3x}$

$$P.I.=\frac{1}{D^3-3D^2+4}e^{3x}$$

$$=\frac{1}{3^3-3.3^2+4}e^{3x}$$

$$=\frac{1}{4}\ e^{3x}$$

$\therefore$ the complete solution is

$$y=c_1\ e^{-x}+(c_2+c_3\ x)\ e^{2x}+\frac{1}{4}e^{3x}$$

Q.19. Solve the equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=2e^{2x}$$

Sol. The equation can be written as

$(D^2+2D+1)\ y=2e^{2x}$

$\Rightarrow$ $(D+1)^2y=2e^x$

The auxilary equation is

$(m+1)^2=0$

$\Rightarrow$ m=–1, –1

$\therefore$ C.F.=$(c_1+c_2x)\ e^{-x}$

P.I.=$\dfrac{1}{(D+1)^2}.2e^{2x}$

$=2\dfrac{1}{(2+1)^2}\ e^{2x}$

$=\dfrac{2}{9}\ e^{2x}$

the complete solution is

$$y=(c_1+c_2x)\ e^{-x}+\frac{2}{9}\ e^{2x}$$

Q.20. Solve the equation

$$\frac{d^2y}{dx^2}-4y-(1+e^x)^2=0$$

Sol. The equation can be written as

$(D^2-4)\ y=(1+e^x)^2$

The auxilary equation is

$m^2-4=0$

$\Rightarrow$ $m^2=4 \Rightarrow m\underline{+}2$

$\therefore$ C.F.=$c_1e^{2x}+c_2e^{-2x}$

P.I.=$\dfrac{1}{D^2-4}(1+e^x)^2$

$$\text{P.I.}=\frac{1}{(D+2)(D-2)}(1+2e^{x}+e^{2x}$$

$$=\frac{1}{(D+2)(D-2)}\,\mathrm{e}^{0x}+2\frac{1}{(D+2)(D-2)}e^{x}$$

$$+\frac{1}{(D+2)(D-2)}e^{2x}$$

$$=\frac{1}{(0+2)(0-2)}e^{0x}+2\frac{1}{(1+2)(1-2)}e^{x}$$

$$+\frac{1}{(2+2)(\mathrm{D}-2)}e^{2x}$$

$$=-\frac{1}{4}+\frac{2}{-3}e^{x}+\frac{1}{4}xe^{2x}$$

$$=-\frac{1}{4}-\frac{2}{3}e^{x}+\frac{1}{4}xe^{2x}$$

$\therefore$ the complete solution is

$$y=c_1e^{2x}+c_1e^{-2x}-\frac{1}{4}-\frac{2}{3}e^{x}+\frac{1}{4}xe^{2x}$$

Preliminary

The study of equations.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where P, Q and R are functions of *x, y, z* play important roles in the theory of **partial differential equations,** and it is essential that they should be understood thoroughly before the study of partial differential equations.

Method of Solution

First Method

We have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$$

If l, m, n are such that $lP + mQ + nR = 0$,

then we get $ldx + mdy + ndz = 0$

If it is an exact differential du (say), then $u = a$ is one part of the complete solution.

Similarly, if we can choose l', m', n' such that

$l'P + m'Q + n'R = 0$

We get $l'dx + m'dy + n'dz = 0$

This gives another integral $v = 6$ on integration. Then the two integrals so obtained form the **solution.**

Second Method

The equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Take any two members $\frac{dx}{P} = \frac{dy}{Q}$ (say) and integrate this equation to obtain an integral $u\ (x, y, z) = a$, where a is an arbitrary constant.

Next choose other two members $\frac{dx}{P} = \frac{dz}{Q}$ (say)

This on integration gives another integral $v(x, y, z) = b$, where b is an arbitrary constant.

The two integrals so obtained form the **complete solution.**

Third Method

When one of the variables is absent from one equation of the set

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

We can **derive** the **integrals** in a simple way. Suppose, for the sake of definitensess, that the equation.

$$\frac{dx}{Q} = \frac{dz}{R}$$

may be written in the form

$$\frac{dy}{dz} = f(y, z)$$

This equation has a solution of the form

$\phi\ (y, z, c_1) = 0$

Solving this equation for z and substituting the value of z so obtained in the equation.

$$\frac{dx}{P} = \frac{dy}{Q}$$

We obtain an ordinary differential equation of type

$$\frac{dy}{dx} = g\ (x, y, c_1) = 0$$

whose solution

$\psi = (x, y, c_1, c_2) = 0$

may readily be obtained

We discuss here a few examples.

Example 1. Solve the equations.

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

Solution. We choose

$l = 1,\ m = 1,\ n = 1$

so that $lP + mQ + nR$

$= (y–z) + (z–x) + (x–y) = 0$

So the function $u\ (x, y, z)$ assumes the form

$du = ldx + mdy + ndz$

$= dx + dy + dz \qquad (\because\ l = m = n = 1)$

$= d\ (x + y + z)$

$\therefore\ u\ (x, y, z) = x + y + z$

Similarly, if we take

$l' = x,\ m' = y,\ n' = z$

so that $l'P + m'Q + n'R$

$= x\ (y–z) + y\ (z–x) + z\ (x–y) = 0$

So the function $v(x, y, z)$ assumes the form

$$d\upsilon = l'\,dx + m'dy + n'dz$$

$$= xdx + ydy + zdz$$

$$= d\,\frac{\left(x^2 + y^2 + z^2\right)}{2}$$

$$\therefore \; v\,(x, y, z) = \frac{x^2 + y^2 + z^2}{2}$$

Hence $x + y + z = a, \qquad x^2 + y^2 + z^2 = b$

form the complete solution (a, b being arbitrary constants)

Example 2. Solve the equations

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Solution. We take first the two members

$$\frac{dx}{yz} = \frac{dy}{zx}$$

which gives $xdx - ydy = 0$ or $d\,(x^2 - y^2) = 0$

This on integration yields $x^2 - y^2 = a$

Similarly, if we take the two members

$$\frac{dy}{zx} = \frac{dy}{xy}$$

we obtain integral $y^2 - z^2 = b$

$\therefore$ the two integrals are

$x^2 - y^2 = a$, $y^2 - z^2 = b$

where a, b are arbitrary constants.

Example 3. Solve the equations

$$\frac{dx}{x+z}=\frac{dy}{y}=\frac{dz}{z+y^2}$$

Solution. We have from

$$\frac{dy}{y}=\frac{dz}{z+y^2}$$

$$\frac{dz}{dy}=\frac{z+y^2}{y}$$

or $\frac{dz}{dy}-y^{-1}z = y$ *(i)*

which is a linear differential equation

This equation has a solution of the form

$$z = \left[c_1+\int y^{e^{-\int\frac{1}{y}dy}}dy\right]\int\frac{1}{y}dy \quad \text{or} \quad \frac{z}{y}=c_1 + y$$

$\therefore$ $z = c_1\, y + y^2$ (*ii*)

Wih this value of z, we have from

$$\frac{dx}{x+z}=\frac{dy}{y}$$

$$\frac{dx}{x+c_1y+y^2}=\frac{dy}{y} \quad \text{or} \quad \frac{dx}{dy}-\frac{1}{y}x = c_1+y \qquad \textit{(iii)}$$

This again is a linear differential equation whose solution is given by

$$x = \left[c_2+\int(c_1+y)^{e^{-\int\frac{1}{y}dy}}\right]e^{\int\frac{1}{y}dy}$$

or

$$\frac{x}{y} = c_2 + \int (c_1+y)\frac{1}{y}dy$$

$$= c_2+c_1 \log y + y$$

or $x = c_2 y + y (c_1 \log y + y)$

Thus the two integrals of the given differential equations are

$z = c_1 y + y^2,\ x = c_2 y + y\ (c_1 \log + y)$

where c_1 and c_2 are arbitrary constants.

EXERCISES

Solve the following differential equations :

1. $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$

2. $\dfrac{dx}{y^2+z^2-x^2} = \dfrac{dy}{-2xy} = \dfrac{dz}{-2xz}$

3. $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$

4. $\dfrac{dx}{x^2(y^3-z^3)} = \dfrac{dy}{y^2(z^3-x^3)} = \dfrac{dz}{z^2(x^3-y^3)}$

5. $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx-lz} = \dfrac{dz}{ly-mx}$

6. $\dfrac{dx}{y^3x-2x^4} = \dfrac{dy}{2y^4-x^3y} = \dfrac{dz}{2z(x^3-y^3)}$

7. $\dfrac{dx}{2xz} = \dfrac{dy}{2yz} = \dfrac{dz}{z^2-x^2-y^2}$

8. $\dfrac{xdx}{z^2-2yz-y^2} = \dfrac{dy}{y+z} = \dfrac{dz}{y-z}$

ANSWERS

1. $x + y + z = a,\ x y z = b$
2. $y = az,\ x^2 + y^2 + z^2 = bz$
3. $x - y = a\ (z-x),\ (x - y)^2\ (x + y + z) = b$
4. $x^2 + y^2 + z^2 = a,\ x^{-1} + y^{-1} + z^{-1} = b$
5. $lx + my + nz = a,\ x^2 + y^2 + z^2 = b$
6. $x^2\ y^2\ z^3 = a,\ (x^3 + y^3)\ z^3 = b$
7. $y = ax,\ x^2 + y^2 + z^2 = bx$
8. $x^2 + y^2 + z^2 = a,\ y^2 - 2yz-z^2 = b$

PARTIAL DIFFERNTIAL EQUATIONS

Partial differential equations arise in **Geometry** and **Physics** when the number of independent variables in the problem under discussion is two or more. However the study is generally confirmed to when z is a function of two independent variables x and y ; we write

$$z = f\ (x,\ y)$$

$\frac{\partial z}{\partial x}$, the partial derivative of z w. r. t. $x,$ is denoted by $p,$ so that $p = \frac{\partial z}{\partial x}$

Similarly $\frac{\partial z}{\partial y}$, the partial derivative of z w. r. t. $y,$ is denoted by $q,$ so that $q = \frac{\partial z}{\partial y}$.

Also the second partial derivative, of z w. r. t. x and y are denoted by $r,\ s$ and $t,$ so that

$$r = \frac{\partial^2 z}{\partial x^2},\quad s = \frac{\partial^2 z}{\partial y\, \partial x},\quad t = \frac{\partial^2 z}{\partial\, y^2}$$

Definition. Partial differential equations, into which two independent variables atleast, and partial derivatives with respect to any or all these variables, may enter.

For Example

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2 \qquad \text{.... (1)}$$

$$\frac{\partial^2 z}{\partial x^2} + (a+b) \frac{\partial^2 z}{\partial y\, \partial x} + ab \frac{\partial^2 z}{\partial y^2} = xy \qquad \text{.... (2)}$$

are partial differential equations.

The order of a partial differential equation is determined by the highest order partial derivative in it. Thus (1) is a partial differential equation of order and (2) is of order 2.

Formation of partial differential equations

Partial differential equations can often be formed

(a) by the elimination of arbitrary constants from a given relation between the variables.

(b) by the elimination of arbitrary functions of the variables.

(a) Elimination of arbitrary constants

We illustrate this the help of examples.

Example 1. Eliminate the arbitrary constants a and c from $x^2 + y^2 + (z - c)^2 = a^2$

Solution. We have the relation

$$x^2 + y^2 + (z - c)^2 = a^2 \qquad \text{..... (1)}$$

Differentiating (1) partially w.r.t. x and y respectively, we obtain

$$2x + 2\ (z - c)\ p = 0 \qquad \text{..... (2)}$$

$$2x + 2\ (z - c)\ q = 0 \qquad \text{..... (3)}$$

$$\left(\because \frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q\right)$$

From (2) and (3) we obtain

$$\frac{x}{y} = \frac{p}{q} \qquad \text{or} \qquad qx - py = 0 \qquad \text{...... (4)}$$

Note :- This is a partial differential equation of order 1. This equation is also *linear i.e.* powers of p and q are both 1.

Example 2. Eliminate the arbitrary constants a and b from $z = (x - a)^2 + (y - b)^2$

Solution. We have the relation

$$z = (x - a)^2 + (y - b)^2 \qquad (1)$$

Differentiating (1) partially w.r.t. x and y respectively, we have

$$\frac{dz}{dx} = 2\ (x - a) \qquad (2)$$

$$\frac{\partial z}{\partial y} = 2\ (y - b) \qquad \text{............(3)}$$

Squaring and adding (2) and (3) we get

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 4\ [(x-a)^2+(y-b)^2]$$

or $\qquad p^2 + q^2 = 4z \qquad$ ....(4)

$[\because\ (x-a)^2 + (y-b)^2 = z]$

Note : Equation (4) is a **partial differential equation** of order 1 but non-linear.

Example 3. Eliminate the arbitrary constants a and b from $z = ax + by + a^2 + b^2$

Solution : We have the relation

$$z = ax + by + a^2 + b^2 \qquad \text{............(1)}$$

Differentiating (1) partially w.r.t. x and y respectively. We get

$$\frac{\partial z}{\partial x} = a \qquad \frac{\partial z}{\partial y} = b$$

or $\quad p = a,\ q = b \qquad$ ....(2)

From (1) and (2) we have

$$z = px + qy + p^2 + q^2 \qquad \text{............(3)}$$

This is **a partial differential equation** of order 1 but **non-linear**. Equation (3) is a partial differential equation in **Clairaut's form.**

Note : In case the member of arbitrary constants are more than two, then three relations namely, the given relation and the two relations obtained by partially differentiating w.r.t. x and y, are **not sufficient** to eliminate these constants. Therefore, in this case we have to take relations involving higher derivatives and the differential equation would not be of order 1. The following example would illustate it.

Example 4. Eliminate the constants $a, b,$ and c from the relation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Solution. We have the relation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \qquad \text{............(1)}$$

Differentiating (1) partially w.r.t. x and y respectively, we have

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \qquad ....(2)$$

and $$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \qquad ....(3)$$

We can re-write (2) and (3) as

$$\frac{x}{a^2}+\frac{z}{c^2}\ p = 0 \qquad ....(2')$$

$$\frac{y}{b^2}+\frac{z}{c^2}\ q = 0 \qquad ....(3')$$

$$\left(\because \frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q\right)$$

There being three constants a, b, c these cannot be eliminated from (1), (2'), (3'). Therefore we need one more relation.

Differentiating (2') again partially w.r.t. $x,$ we get

$$\frac{1}{a^2}+\frac{p^2}{c^2}+\frac{z}{c^2}r = 0 \qquad (4)$$

Multiplying it by x and subtracting (2') from it, we get

$$\frac{1}{c^2}\ [xp^2+xzr - pz] = 0$$

or $$pz = xp^2 + xzr \qquad (5)$$

This is the partial differential equation obtained after eliminating a, b, c and is of order 2 $\left(\because r = \frac{\partial^2 z}{\partial x^2}\right)$

Note : Other partial differential equations can also be obtained; for example, if we differentiate (3') partially w.r.t. y, we get

$$\frac{1}{b^2}+\frac{q^2}{c^2}+\frac{z}{c^2}\ t = 0$$

Multiplying it by y and subtracting (3') from it, we get

$$\frac{1}{c^2}[y\ q^2+yzt-zq] = 0$$

or $\quad q\ z = yq^2 + y\ z\ t$ ....(6)

This again is the partial differential equation obtained after eliminating a, b, c and is of order 2. $\left(\because t = \frac{\partial^2 z}{\partial x^2}\right)$

(6) Eliminating of arbitrary functions.

Consider a relation.

$$\text{F}\ (u,\ v) = 0 \qquad \text{............(1)}$$

where u and v are known functions of x, y, z and F is an **arbitrary functions** of u and v.

We shall show that elimination of F leads to the partial differential equation of the form Pp + Qq = R where P, Q and R are functions of x, y and z.

If we differentiate equation (1) w.r.t. x and y respectively, then

$$\frac{\partial \text{F}}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial u}p\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right) = 0 \qquad \text{............(2)}$$

$$\frac{\partial \text{F}}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right) = 0 \qquad \text{............(3)}$$

Eliminating $\frac{\partial \text{F}}{\partial u}$ and $\frac{\partial \text{F}}{\partial v}$ from equations (2) and (3), we obtain

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q \end{vmatrix} = 0$$

Expanding this determinant we obtain

$$\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)+p\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right) + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right) \text{q} = 0$$

or $\quad Pp + Qq = R \qquad$ ..... (4)

where

$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Example 1. Eliminate F from F $(u, v) = 0$, where $u = lx + my + nz$, $v = x^2 + y^2 + z^2$

Solution. The elimination of F leads to a partial differential equation of the form.

$$Pp + Qq = R \qquad ....(1)$$

where

$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$$

$$= m\ .\ 2z - n.\ 2y = 2\ (mz - ny)$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$$

$$= n\ .\ 2x - 1\ .\ 2z = 2\ (nx - 1z)$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$= l.2\,y - m.2x = 2\,(ly - mx)$$

With these values of P, Q and R we have from (1)

$$(mz - ny)\,p + (nx - lz)\,q = ly - mx$$

which is the required **partial differential equation** of order 1. This is **linear** also.

Example 2. Eliminate F from F $(u, v) = 0$ where

$$u = x^2 + y^2 + z^2, \quad v = z^2 - 2x\,y$$

Solution. The elimination of F leads to a partial differential equation of the form

$$Pp + Qq = R \qquad \text{.............(1)}$$

where

$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$$

$$= 2y\,.\,2z - 2z\,(-2x) = 4\,(y+x)\,z$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$$

$$= 2z\,(-2y) - 2x\,(2z) = -4\,(y + x)\,z$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$= 2x\,(-2x) - 2y\,(-2y) = -4\,(x^2-y^2)$$

With these values of P, Q and R we have from (1)

$$z\,(x + y)\,p - z\,(x + y)\,q = y^2 - x^2$$

which is the required **partial differential equation of order 1.** This is linear also.

Linear partial differential equations of the first order.

These are partial differential equations of the form

$$Pp + Qq = R \qquad \text{.............(1)}$$

where P, Q and R are given functions of x, y and z [which do not involve p or q], p denotes $\frac{\partial z}{\partial x}$, q denotes $\frac{\partial z}{\partial y}$ and we wish to find a relation between x, y and z involving an **arbitrary function.**

The first systematically theory of equations of this type was given by Lagrange's. For that reason equation (1) is frequently referred to as Lagrange's equation.

Lagrange's reduced the problem of finding the **general solution** of (1) to that of solving an **auxiliary system** (called the Lagrange system) of ordinary differential equations.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{............(2)}$$

by showing that

$$F\,(u, v) = 0 \text{ (F, arbitrary)} \qquad \text{............(3)}$$

is the general solution of (1) provided

$$u\ (x, y, z) = a$$

$$v\ (x, y, z) = b \qquad \text{............(4)}$$

are two **independent solutions** of (2)

Here a and b are arbitrary constants and atleast one of u, v must contain z.

Example 1. Find the general solution of the partial differential equation $x^2p + y^2q = (x + y)\, z$

Solution. The given partial differential equation is

$$x^2p + y^2q = (x + y)\, z \qquad \text{............(1)}$$

The auxiliary system is

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{(x+y)z}$$

Taking $\frac{dx}{x^2} = \frac{dy}{y^2}$ we obtain

$$x^{-1} - y^{-1} = a \qquad \text{............(3)}$$

Also $\frac{dz}{(x+y)z} = \frac{dx - dy}{x^2 - y^2}$

or $\quad \frac{dz}{z} = \frac{dx - dy}{x^2 - y^2}$

Integrating we obtain

$$\frac{x-y}{z} = b \qquad \text{............(4)}$$

Combining (3) and (4) we obtain

$$\frac{1}{x} - \frac{1}{y} = a \qquad \frac{x-y}{z} = b$$

or $\quad \frac{y-x}{xy} = a$

or $\quad \frac{z}{xy} = -\frac{a}{b} \qquad (\because\ x-y=bz)$ or

or $\quad \frac{xy}{z} = -\frac{a}{b}$

$= c$, say

$\because$ The general solution is

$$F\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0 \qquad \text{............(5)}$$

where the function F is arbitrary.

Check. Here $u\ (x,\ y,\ z) = \frac{xy}{z}$

$v\ (x,\ y,\ z) = \frac{x-y}{z}$

$y - x = -bz - \frac{bz}{xy} = \text{a}$

The elimination of F from (5) leads to a partial differential equation of the form

$$Pp + Qq = R$$

where

$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial v}{\partial z}\frac{\partial v}{\partial y}$$

$$= -\frac{x}{z}\left(\frac{x-y}{z^2}\right) - \left(\frac{xy}{z^2}\right)\left(-\frac{1}{z}\right) = -\frac{x^2}{z^3}$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial z}$$

$$= \left(-\frac{xy}{z^2}\right)\left(\frac{1}{z}\right) - \left(\frac{y}{z}\right)\left(-\frac{x-y}{z^2}\right) = \frac{y^2}{z^3}$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$= \left(\frac{y}{z}\right)\left(-\frac{1}{z}\right) - \left(\frac{x}{z}\right)\left(\frac{1}{z}\right) = \frac{(x+y)}{z^2}$$

Substituting these values in Pp+Qq = R, we obtain

$$p\left(-\frac{x^2}{z^3}\right) + q\left(\frac{y^2}{z^3}\right) = \left(-\frac{(x+y)}{z^2}\right)$$

$$px^2 + qy^2 = (x + y)\, z$$

which is the required partial differential equation.

Example 2. Solve $(y + z)\, p + (z + x)\, q = x + y$

Solution. The given partial differential equation is

$$(y + z)\, p + (z + x)\, q = x + y \qquad \text{.............(1)}$$

Here P = $y + z$, Q = $z + x$, R = $x + y$

The auxiliary system is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

or $$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \qquad \text{............(2)}$$

Each ratio = $$\frac{dx-dy}{(y+z)-(z+x)} = \frac{dy-dz}{(z+x)-(x+y)}$$

or $$\frac{dy-dx}{y-x} = \frac{dz-dy}{z-x}$$

or $$\frac{d(y-x)}{(y-x)} = \frac{d(z-y)}{(z-y)}$$

Integration yields $\frac{y-x}{z-y}$ = a

Also, each ratio = $$\frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}$$

or $$\frac{d(x+y+z)}{(x+y+z)} + 2\,\frac{d(z-y)}{(z-y)} = 0$$

Integration yields

$$(x + y + z)\,(z - y)^2 = b \qquad \text{............(4)}$$

Hence the general solution of (1) is

$$F\left(\frac{y-x}{z-y}, (x+y+z)(z-y)^2\right) = 0$$

where F is arbitrary.

Example 3. Solve

$(mz - ny)\, p + (nx - lz)\, q = ly - mx$

Solution. The given partial differential equation is

$$(mz - ny)\, p + (nx - lz)\, q = ly - mx \qquad \text{.........(1)}$$

Compare it with Pp + Qq = R, we have

P = $mz - ny$, Q = $nx - lz$, R = $ly - mx$

The auxiliary system is

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx} \qquad \text{.............(2)}$$

We choose $l = l, m = m, n = n$

So that

lP + mQ + nR

$$= l\ (mz - ny) + m\ (nx-lz) + n\ (ly-mx) = 0$$

So the function u (x, y, z) assumes the form

$$du = ldx + mdy + ndz$$

$$= d\ (l\ x + my + nz)$$

$$\therefore \quad u\ (x, y, z) = lx + my + nz \qquad \text{.............(3)}$$

Similarly, if we take $l' = x, m' = y, n' = z$

So that

l' P + m' Q + n' R

$$= x\ (mz - ny) + y\ (nx - lz) + z\ (ly - mx) = 0$$

So the function $v(x, y, z)$ assumes the form

$$dv = xdx + ydy + zdz$$

$$= d\left(\frac{x^2+y^2+z^2}{2}\right)$$

$$\therefore \quad v\ (x, y, z) = \frac{x^2+y^2+z^2}{2} \qquad \text{.............(4)}$$

Hence the general solution of (1) is

F (u, v) = 0, F being arbitrary

where u, v are given by (3) and (4)

Alternative method. To find two integrals

$u\ (x, y, z) = a$ and $v\ (x, y, z) = b$ we proceed as follow.

The auxiliary system is

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Each ratio = $\dfrac{ldx+mdy+ndz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}$

or $ldx + mdy + ndz = 0$

Integration yields $lx + my + nz = a$ (1)

Further, each ratio $=\dfrac{xdx+ydy+zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}$

or $xdx + ydy + zdz = 0$

Again integration yields $x^2 + y^2 + z^2 = b$ (2)

Hence the general solution is

$$F(l\,x + my + nz,\ x^2 + y^2 + z^2) = 0 \qquad (3)$$

where F is arbitrary.

Generalization. The general solution of the equation.

$$P_1p_1 + P_2p_2 + ... + P_n\,p_n = R$$

$\therefore$ $F(u_1, u_2,.....u_n) = 0,$

where

$u_1 = c_1$, $u_2 = c_2$,........., $u_n = c_n$ are any n independent solutions of the **auxiliary system.**

$$\frac{dx_1}{P_1}=\frac{dx_2}{P_2}=.....=\frac{dx_n}{P_n}=\frac{d_z}{R}$$

Note : Besides the general integral, special integrals exist, for exceptional equations.

Example 1. If z is a function of x_1, x_2 and x_3 which satisfies the partial differential equation.

$$(x_2 - x_3)\frac{\partial z}{\partial x_1} + (x_3 - x_1)\frac{\partial z}{\partial x_2} + (x_1 - x_2)\frac{\partial z}{\partial x_3} = 0$$

show that z contains x_1, x_2 and x_3 only in combinations $x_1 + x_2 + x_3$ and

$x^2_1+x^2_2+x^2_3$

Solution. The given partial differential equation is $(x_2 - x_3)\, p_1 + (x_3 - x_1)\, p_2 + (x_1 - x_2)\, p_3 = 0,$

where $p_1 = \dfrac{\partial z}{\partial x_1}$, $p_2 = \dfrac{\partial z}{\partial x_2}$, $p_3 = \dfrac{\partial z}{\partial x_3}$

The auxiliary equations are

$$\frac{dx_1}{x_2 - x_3} = \frac{dx_2}{x_3 - x_1} = \frac{dx_3}{x_1 - x_2} = \frac{dz}{0}$$

These are equivalent to three relations. $dz = 0$ (1)

$dx_1 + dx_2 + dx_3 = 0$ (2)

$x_1\, dx_1 + x_2\, dx_2 + x_3\, dx_3 = 0$ (3)

which show that the integrals are

$z = c_1,\ x_1 + x_2 + x_3 = c_2,\ x^2_1 + x^2_2 + x^2_3) = c_3$

Hence the general solution is

$F(z_1 x_1 + x_2 + x_3,\ x^2_1 + x^2_2 + x^2_3) = 0$

or $z = f\,(x_1 + x_2 + x_3,\ x^2_1 + x^2_2 + x^2_3)$

where F and f being arbitrary.

EXERCISE

Find the general integrals of the following equations :

1. $xp + yq = z$
2. $yzp + zxq = xy$
3. $z\,(xp - yq) = y^2 - x^2$
4. $px\,(x + y) - qy\,(x + y) = -\,(x - y)\,(2x + 2y + z)$
5. $zp - zq = z^2 + (y + x)^2$
6. $-p_1 + p_2 + p_3 = 1$
7. $(x_3 - x_2)\, p_1 + x_2\, p_2 - x_3 p_3 = x_2\,(x_1 + x_3) - x^2_2$

ANSWERS

1. $F\left(\frac{x}{y}, \frac{y}{z}\right) = 0$

2. $F(y^2 - x^2, z^2 - x^2) = 0$

3. $F(xy, x^2 + y^2 + z^2) = 0$

4. $F[xy, (x + y)(x + y + z)] = 0$

5. $F[y + x \log(z^2 + y^2 + 2yx + x^2) - zx] = 0$

6. $F[z + x_1, x_1 + x_2, x_2 + x_3] = 0$

7. $F(z - x_1 x_2, x_1 + x_2 + x_3, x_2 x_3) = 0$

NON LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

We now turn to more **difficult** problem of finding the solutions of the partial differential equation.

$$f(x, y, z, p, q) = 0 \qquad (1)$$

in which the function f is not necessarily linear in p and q.

We saw (in the formulation of partial differential equations) that the partial differential equations of the two parameter system.

$$f(x, y, z, a, b) = 0 \qquad (2)$$

was of this form. It will now be **shown** (Charpit's method) that the **converse** is also true *i.e.,* that any partial differential equation of the type (1) has solutions of the type (2).

Classification of integrals

Before trying to find an integral of a partial differential equation, we propose the different forms of integrals. For convenience, we shall take only two independent variables.

(a) The complete integral

Let the relation containing x, y and z be

$$f(x, y, z, a, b) = 0 \qquad (1)$$

By elimination of **arbitrary constants** we can find a partial differential equation.

$$f(x, y, z, p, q) = 0 \qquad (2)$$

Obtaining (1) when (2) is given is known as solving (2) [The methods of doing so will be discussed later].

When the relation (1) has as many arbitrary constants as there are **independent**

variables in (2), it is known as the **complete integral** of (2).

If particular values be given to the arbitrary constants, the complete integral becomes a **particular integral.**

(b) Singular integral.

The singular integral can be obtained by eliminating a and b from the relations $F(x, y, z, a, b) = 0, \dfrac{\partial F}{\partial a} = 0, \dfrac{\partial F}{\partial b} = 0$. This integral does not contain any arbitrary constant, neither can it be obtained from the complete integral by giving particular values to the constants. This solution represents the envelope of the surface represented by (1).

(c) The General integral

If a and b are functionally related *i.e.,* $b = \phi(a)$. Then (1) becomes.

$$F[x, y, z, a, \phi(a)] = 0$$

Eliminating a between this equation and $\dfrac{\partial F}{\partial a} = 0$, we get the **general integral**. It does not contain any arbitrary consant, but it is different from the singular integral, neither can it be obtained from the **complete integral** by giving different values to the constants.

Charpit's method.

Let the partial differential equation be

$$f(x, y, z, p, q) = 0 \qquad (1)$$

The fundamental idea in Charpit' method is the introduction of a second partial differential equation of the first order.

$$g(x, y, z, p, q, a) = 0 \qquad (2)$$

which contains an **arbitrary constant** a and which is such that

(a) Equations (1) and (2) can be solved to give

$$p = p(x, y, z, a), \; q = q(x, y, z, a)$$

(b) The equation

$$dz = p(x, y, z, a)\, dx + q(x, y, z, a)\, dy \qquad (3)$$

is integrable. When such a function g has been found, the solution of equation (3)

$$F(x, y, z, a, b) = 0 \qquad (4)$$

containing two arbitrary constants *a, b* will be a solution of equation (1)

The **main problem** is the determination of the second equation (2).

Determination of *g (x, y, z, p, q, a)*

Differentiate (1) and (2) w.r.t. x and y separately, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad (5)$$

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}p + \frac{\partial g}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial x} = 0 \qquad (6)$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad (7)$$

$$\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}q + \frac{\partial g}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad (8)$$

Multiply (5) by $\frac{\partial g}{\partial p}$, (6) by $\frac{\partial f}{\partial p}$ and subtracting to eliminate $\frac{\partial p}{\partial x}$, we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}\right) + p\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial f}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \qquad (9)$$

and multiplying (7) by $\frac{\partial g}{\partial q}$, (8) by $\frac{\partial f}{\partial q}$ and subtracting to eliminate $\frac{\partial q}{\partial y}$

we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial q}\right) + p\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial q}\right)\frac{\partial f}{\partial y} = 0 \qquad (10)$$

Now in the functions we shall come across

$$\frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial p}{\partial y}$$

and so adding (9) and (10) and rearranging, we have

$$\left(-\frac{\partial f}{\partial p}\right)\frac{\partial g}{dx}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial g}{dy}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial z}$$

$$+\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{dz}\right)\frac{\partial g}{\partial p}+\left(\frac{\partial f}{dy}+q\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial q}=0 \qquad (11)$$

which may be regarded as a **linear partial differential** of the first order to determine g.

The corresponding auxiliary system is

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{\frac{-\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}$$

$$= \frac{dp}{\frac{\partial f}{dx}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dg}{0} \qquad (12)$$

Note : It is evident that these equations can hold only if $dg = 0$ *i.e.,* $g = a$ an arbitrary constant. If any integral of equations (12) can be found involving p or q or both, the integral may be taken as the additional partial differential equation (2) which is **conjuction** with (1) will give values of p and q to make (3) integrals. This will give a complete integral (4) of (1), from which general and singular integrals can be deduced in the usual way.

The auxiliary (or subsidiary) equations are often called Charpit's equations.

Special type of first order equations.

Type 1. Equations involving only *p* and *q*

For equations of the type $f(p, q) = 0$, we have

$$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0, \frac{\partial f}{\partial z} = 0$$

Charpit's equations reduce to

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dp}{0} = \frac{dq}{0}$$

An **obvious solution** of these equations is

$p = a$

the corresponding value of q being obtained from $f(p, q) = 0) = 0)$ in the form $f(a, q) = 0$ so that $q = Q(a)$, a constant.

The **solution** of the equation $f(p, q) = 0$ is

$z = ax + Q(a)\,y + b$

where b is an arbitrary constant.

Note : We have chosen the equation $dp = 0$ to provide our second solution. In some problems the amount of computation involved is considerably reduced if we take instead $dq = 0$, leading to $q = a$.

Example 1. Find a complete integral of the equation

$p + q = pq$

Solution. The given equation is of the type

$f(p, q) = 0$ ....(1)

where $f(p, q) = p + q - pq$

Charpit's equations give

$p = a$ ....(2)

Solving (1) and (2) we obtain

$a + b = aq$

or $q = \frac{a}{a-1}$ ....(3)

Substituting the values of p and q in

$dz = pdx + qdy$ ....(4)

so that (4) is integrable. We get

$$z = ax + \frac{a}{a-1} y + b$$

where a and b are arbitrary constants.

Type. 2. Equations not involving the independent variables.

If the partial differential equation is of the form

$$f(z, p, q) = 0 \qquad \text{............(1)}$$

we have $\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0$

Charpit's equations reduce to

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{\frac{-\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dp}{p\frac{\partial f}{dz}} = \frac{dq}{q\frac{\partial f}{dz}} \qquad \text{............(2)}$$

the last two of which leads to the relation.

$$p = aq \qquad \text{............(3)}$$

Solving (1) and (3) we obtain expressions for p and q from which a **complete integral** follows immediately.

Example 2. Find complete integral of the equation.

$$z = p^2 - q^2$$

Solution. The given equation is of the type

$$f(z, p, q) = 0 \qquad \text{............(1)}$$

where $f(z, p, q) = z - p^2 + q^2$

Charpit's equations give

$$p = aq \qquad \text{............(2)}$$

Solving (1) and (2) we have

$$z = a^2q^2 - q^2$$

or $\quad q = \pm \left(\dfrac{z}{a^2 - 1} \right)^{\frac{1}{2}}$ ....(3)

Substitute the values of p and q in

$dz = pdx + qdy$

we have

$dz = (adx + dy)\ q \ (\because p = aq)$

or $\quad \pm (a^2-1)^{\frac{1}{2}} z^{-\frac{1}{2}} d^2 = d\ (ax + y)$

or $\quad \left[\pm 2 \left(a^2 - 1\right)^{\frac{1}{2}} z^{\frac{1}{2}} \right] = d\ (ax + y)$

Integrating we obtain

$\pm 2\ (a^2-1)^{\frac{1}{2}} z^{\frac{1}{2}} = ax + y + b$

or $\quad 4\ (a^2 - 1)\ z = (ax + y + b)^2$

which is the complete integral.

Type 3. Separable equations.

We say that a first order partial differential equation is separable if it can be written in the form

$f\ (x, p) = g\ (y, q)$ ....(1)

Charpit's equations reduce to

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{\frac{\partial y}{\partial q}} \qquad \text{F}\ (x, y, z, p, q) = f(x, p) - g(y, q)$$

$$= \frac{dz}{-p\frac{\partial f}{\partial p} + q\frac{\partial q}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x}} \qquad \frac{\partial \text{F}}{\partial x} = \frac{\partial f}{\partial x}, \frac{\partial \text{F}}{\partial y} = -\frac{\partial g}{\partial y}$$

$$= \frac{dq}{-\frac{\partial g}{\partial y}} \qquad \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial p} = \frac{\partial f}{\partial p}, \frac{\partial F}{\partial p} = -\frac{\partial g}{\partial q}$$

From the equation $\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dp}{\frac{\partial f}{\partial x}}$ we obtain

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial p} dp = 0$$

or $d\,[f\,(x,\ p)] = 0$

Integration yields

$$f\,(x,\ p) = a \qquad \text{............(2)}$$

Hence we **determine** $p,\ q$ from the relations

$$f\,(x,\ p) = a,\ g\,(y,\ q) = a$$

and then proceed as in the general theory.

Example 3. Find complete integral of the equation

$$p^2q^2+x^2y^2 = x^2p^2\ (x^2+y^2)$$

Solution. The given partial differential equation is

$$p^2q^2 + x^2y^2 = x^2q^2\ (x^2 + y^2)$$

or $\frac{p^2}{x^2}\ q^2 + y^2 = q^2\ (x^2 + y^2)$

or $\frac{p^2}{x^2} + \frac{y^2}{q^2} = x^2 + y^2$

or $\frac{p^2}{x^2} - x^2 = y^2 - \frac{y^2}{q^2}$

This is the form $f\,(x,\ p) = g\,(y,\ q)$, where

$$f(x,\ p) \equiv \frac{p^2}{x^2} - x^2 \text{ and } g(y,\ q) = y^2 - \frac{y^2}{q^2}$$

Here Charpit's equation yield

$$\frac{p^2}{x^2} - x^2 = a^2 \qquad\qquad y^2 - \frac{y^2}{q^2} = a^2$$

$$\text{or} \quad p = x\,(x^2+a^2)^{\frac{1}{2}} \qquad \text{or} \qquad q = \frac{y}{\left(y^2 - a^2\right)^{\frac{1}{2}}}$$

When we substitute the values of p and q in $dz = pdx + qdy$ we obtain

$$dz = x\,(x^2+a^2)^{\frac{1}{2}}\ dx + y\,(y^2-a^2)^{\frac{1}{2}}\ dy$$

Integration yields

$$z = \frac{1}{2}\ \frac{\left(x^2+a^2\right)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2}\frac{\left(y^2-a^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + b$$

$$\text{or} \quad z = \frac{1}{3}\ (x^2+a^2)^{\frac{3}{2}} + (y^2-a^2)^{\frac{1}{2}} + b$$

which is the complete integral.

Type 4. Clairaut's equations.

A first-order partial different equation is said to be of Clairaut's form if it can be written as

$$z = pz + qy + f(p,\ q) \qquad ....(1)$$

Charpit's equations reduce to

$$\frac{dx}{-\frac{\partial f}{\partial p} - x} = \frac{dy}{-\frac{\partial f}{\partial q} - y} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q} - px - qy}$$

$$= \frac{dp}{p-p} = \frac{dp}{q-q}$$

From the last two equations we obtain

$p = a, \ q = b$

If we put these values in (1), we get the complete integral

$$z = ax + by + f(a, b) \qquad \text{.............(2)}$$

Example 4. Find a complete integral of the equation

$(p + q)(z - px - qy) = 1$

Solution. We write the given equation in the form

$$z - px - qy = \frac{1}{p+q}$$

or $\quad z = px + qy + \dfrac{1}{p+q}$ [Clairaut's form]

$\therefore$ The complete integral is

$$z = ax + by + \frac{1}{a+b}$$

where a, b are arbitrary constants.

Example 5. Find the complete integral of the equation.

$pqz = p^2 (xp + p^2) + q^2 (py + q^2)$

Solution. We write the given equation in the form

$pqz = p^2qx + q^2py + p^4 + q^4$

or $\quad z = px + qy + \dfrac{p^4+q^4}{pq}$ [Clairaut's form]

$\therefore$ The complete integral is

$$z = ax + by + \frac{a^4+b^4}{ab}$$

where a, b are arbitrary constants.

Example 6. Solve $z^2 (p^2 + q^2) = x^2 + y^2$

Solution. The given partial differential equation is

$$z^2 (p^2 + q^2) = x^2 + y^2 \qquad ....(1)$$

Note : This is not of the **standard form** but can be transformed into one. We can write has above equation as

$$\left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2 \qquad ....(2)$$

Let us put $2\,dz = dZ$, then $Z = \dfrac{z^2}{2}$

Now $\dfrac{\partial Z}{\partial x} = \dfrac{\partial Z}{\partial z}\dfrac{\partial z}{\partial x} = z\dfrac{\partial z}{\partial x} = P$, say

$\dfrac{\partial Z}{\partial y} = \dfrac{\partial Z}{\partial z}\dfrac{\partial Z}{\partial y} = z\dfrac{\partial z}{\partial y} = Q$, say

The equation (2) now becomes

$P^2 + Q^2 = x^2 + y^2$,

with Z as the dependent variable

Note : This equation is of Type 3 *viz*

$P^2 - x^2 = y^2 - Q^2$

Charbit's equations give

$P^2 - x^2 = a^2 \qquad y^2 - Q^2 = a^2$

then $P = (x^2+a^2)^{\frac{1}{2}}$ and $Q = (y^2-a^2)^{\frac{1}{2}}$

$\therefore \quad dZ = P\,dx + Q\,dy$

given $dZ = (x^2+a^2)^{\frac{1}{2}}\,dx + (y^2-a^2)^{\frac{1}{2}}\,dy$

the integral of which is

$$Z = \frac{x}{2}(x^2+a^2)^{\frac{1}{2}} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right) + \frac{y}{2}(y^2-a^2)^{\frac{1}{2}} - \frac{a^2}{2}\log\left(y+\sqrt{y^2+a^2}\right) + b$$

or $z^2 = x(x^2+a^2)^{\frac{1}{2}}+y(y^2-a^2)^{\frac{1}{2}}+a^2 \log\left(\frac{x+\sqrt{x^2+a^2}}{y+\sqrt{y^2-a^2}}\right)+b$

which is the complete integral of the given equation.

EXERCISES

Find complete integrals of the equations.

1. $pq = 1$
2. $p^2z^2+q^2 = 1$
3. $p^3y\ (1+x^2) = qx^2$
4. $z^2 = pq\ xy$
5. $p^2x + q^2y = z$
6. $(p^2+q^2)\ y = qz$
7. $2\ (z + px + qy) = yp^2$

ANSWERS

1. $z = ax + \frac{y}{a} + b$
2. $ax\ (1+a^2z)^{\frac{1}{2}}-\log\left[az+(1+a^2z^2)^{\frac{1}{2}}\right]$

   $= 2a\ (ax + y + b)$
3. $z = a\sqrt{1+x^2} + \frac{a^2}{2}y^2 + b$
4. $z = b\ x^a\ y^{\frac{1}{a}}$
5. $[(1+a)z]^{\frac{1}{2}} = (ax)^{\frac{1}{2}} + y^{\frac{1}{2}}+b$
6. $az^2 = (x + b)^2 + y^2$
7. $z = \frac{ax}{y^2}-\frac{a^2}{4y^3}+\frac{b}{y}$

PARTIAL DIFFERENTIAL EQUATIONS OF SECOND & THIRD ORDERS WITH CONSTANT COEFFICIENTS

(Homogeneous and Non-homogeneous equations)

A partial differential equation which contains derivatives of any order but none of the derivatives appears in degree higher than unity is called a linear partial differential equation.

If the **coefficients** of various terms are constants then it is called the linear partial differential equation with **constant coefficients.**

Homogeneous linear partial differential equation with constant coefficients.

If the derivatives in a linear partial differential equation are of the same order, then the linear partial differential equation is said to be homogeneous otherwise non-homogeneous. For example,

$$A\frac{\partial^2 Z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} = 2x + 3y \qquad \text{..... (1)}$$

in which A, B, C are constants, is a **homogeneous linear partial differential equation with constant coefficients, of order two, while**.

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial y^2} + C\frac{\partial^2 z}{\partial y^2} + L\frac{\partial z}{\partial x} + M\frac{\partial z}{\partial y} + Nz = 2x + 3y$$

..... (2)

is a non-homogeneous linear partial differential equation with constant coefficient, the order still being two.

Homogeneous equation with constant coefficients Method of solution.

The equation is of the form

$$F(D, D')z = f(x, y), \; D=\frac{\partial}{\partial x}, \; D'=\frac{\partial}{\partial y}$$

where F is a **homogeneous function** of the derivatives and the coefficients of all the derivatives are constants.

As in ordinary differential equations, the solution is composed of two distinct steps.

(i) First find the solution of

$F(D, D')\, z = 0$

This is known as the **complementary** function.

(ii) Next find the particular integral (P.I.) from

$$z = \frac{1}{F(D,D')} f(x, y)$$

The **general solution** is the sum of these two results.

Note : If $f(x, y)=0$, then the first result is the **general solution.**

To find the Complementary Function

The General solution of

$$Ap + Bq = 0, \qquad p = \frac{\partial z}{\partial x}, \; q = \frac{\partial z}{\partial y}$$

$$\therefore \quad z = \phi\left(y - \frac{B}{A}x\right) = \phi\,(y + mx)$$

where ϕ is an arbitrary function. See how.

The given partial differential equation is

$Ap + Bq = 0$, which is of Lagrange's form.

The auxiliary system is

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{0}$$

First two relations give $dy - \frac{B}{A}dx = 0$

or $\quad y - \frac{B}{A}x = a$

Last relation gives $dz = 0$ or $z = b$

Hence **general solution** is

$\phi\,(a,\ b) = 0$ or $b = \phi\,(a)$

which gives $\quad z = \phi\left(y - \frac{B}{A}x\right)$

or $z = \phi\,(y + mx)$, say $\left(m = -\frac{B}{A}\right)$

where ϕ is an arbitrary function.

Let us now suppose that

$z = \phi\,(y+mx) = \phi(u)$, ϕ being arbitrary is a solution of the equation.

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x\,\partial y} + C\frac{\partial^2 z}{\partial y^2} = 0$$

Now $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = m\frac{\partial \phi}{\partial u}$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = \frac{\partial \phi}{\partial u}$$

Therefore substituting in the equation, we have

$$\left(\frac{\partial \phi}{\partial u}\right)^2 (Am^2 + Bm + C) = 0$$

Since ϕ is arbitrary, $\frac{\partial\phi}{\partial u} \neq 0$ identically, and hence

$$Am^2 + Bm + C = 0 \qquad (*)$$

Note : that the auxiliary equation is simply obtained by putting $\frac{\partial z}{\partial x} = m$, $\frac{\partial z}{\partial y} = 1$ in F (D, D') = 0

Suppose (*) has two real and unequal roots m_1 and m_2, then

$z = \phi_1(y+m_1x)$ and $z = \phi_2(y + m_2x)$

are two distinct solutions of the equation and hence

$z = \phi_1 (y + m_1x) + \phi_2 (y + m_2x)$ is also a solution.

Since this contains two arbitrary functions hence it is the **general solution,** which in this case is identical with the **complementary function.**

Example 1. Solve

$$(D^2–DD'–2^2D')\ z = 0$$

Solution : Let $z = (y + mx)$ be a solution of the given equation where m is given by the auxiliary equation

$m^2–m–2 = 0$ which give $m = 2, –1$

Hence the general solution is

$z = \phi_1(y + 2x) + \phi_2 (y - x)$

where ϕ_1 and ϕ_2 being arbitrary.

Example 2. Solve $(D^3 - 7DD^{2'} + 6^3D')\ z = 0$

Let $z = (y + mx)$ be a solution of the given equation where m is given by the auxiliary equation.

$m^3 - 7m + 6 = 0$

Solving we obtain $m = 1, 2, –3$

$\therefore$ the general solution is

$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y - 3x)$

where ϕ_1, ϕ_2 and ϕ_3 being arbitrary.

Example 3. Solve $(D^2 + a^2D^{2'})\ z = 0$

Solution : The given partial differential equation is

$(D^2 + a^2 D^{2'}) z = 0$

Let $z = \phi (y + mx)$ be a solution of the given equation where m is given by the auxiliary equation.

$m^2 + a^2 = 0,$ or $m = i\, a, - ia$

$\therefore$ the general solution is

$z = \phi_1 (y + iax) + \phi_2 (y - iax)$

where ϕ_1 and ϕ_2 being arbitrary

If, however, some of the roots of the auxiliary equation be equal, say $m_1 = m_2$

then the general solution of $(D{-}mD')^2 z = 0$ is

$z = x\phi_1 (y + mx) + \phi_2 (y + mx)$

where ϕ_1 and ϕ_2 being arbitrary. See how

We have

$(D{-}mD')\ (D{-}mD')\ z = 0$ (1)

Putting $(D{-}mD')z = u$ (2)

then equation (1) becomes

$(D - mD')\ u = 0$

and its solution is $u = \phi_1 (y + mx)$

Putting the value of u in (2) we have

$(D - mD')\ z = \phi_1(y + mx)$

which is of the form $Pp + Qq = R$

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi_1(y+mx)}$$

The first two relations give $dy{+}mdx{=}0$ or $y{+}mx{=}a$

Again from the relations $\frac{dx}{1} = \frac{dz}{\phi_1(y+mx)}$

we have $dz = \phi_1 (a)\ dx$ $(\because\ y + mx = a)$

or $z = x\ \phi_1(a) + b$ or $z - x\phi_1(y + mx) = b$

$\therefore$ the general solution of (1) is

$z - x\ \phi_1\ (y + mx) = \phi_2\ (y = mx)$

or $\quad z = x\ \phi_1\ (y + mx) + \phi_2\ (y + mx)$

where ϕ_1 and ϕ_2 being arbitrary.

Example 4. Solve $(D^3 - 3DD^{2'} + 2D^{3'})\ z = 0$

Solution. The given partial differential equation is

$$(D^3 - 3DD^{2'} + 2D^{3'})\ z = 0$$

Let $z = \phi(y + mx)$ be the solution of the above equation then m is given by the **auxiliary equation**.

$$m^3 - 3m + 2 = 0$$

Solving we obtain $m = 1, 1, -2$

$\therefore$ the general solution is

$$z = x\ \phi_1(y + x) + \phi_2\ (y + x) + \phi_3\ (y - 2x)$$

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary.

EXERCISE

Solve the following partial differential equations.

1. $(D^2 - 3a\ DD' + 2a^2\ D^{2'})\ z = 0$
2. $(2D^2D' - 3DD'^2 + D'^3)\ z = 0$
3. $(2D^2 + 5DD' + 2D'^2)\ z = 0$
4. $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)\ z = 0$
5. $(25D^2 - 40DD' + 16D'^2)\ z = 0$
6. $(D^2 - 4DD' + 4D'^2)\ z = 0$
7. $(D^3 - 2D^2D' + DD'^2)\ z = 0$
8. $(D^3 - 3D^2D' + 3DD'^2 - D'^3)\ z = 0$

ANSWERS

1. $z = \phi_1(y + ax) + \phi_2(y + 2ax)$
2. $z = \phi_1(y) + \phi_2(x + y) + \phi_3(x + 2y)$
3. $z = \phi_1(2y - x) + \phi_2(y - 2x)$

4. $z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$

5. $z = \phi_1(5y + 4x) + x\,\phi_2(5y + 4x)$

6. $z = \phi_1(y+2x) + x\,\phi_2(y + 2x)$

7. $z = \phi_1(y) + \phi_2(y + x) + x\,\phi_3(y + x)$

8. $z = \phi_1(y + x) + x\,\phi_2\,(y + x) + x^2\,\phi_3(y + x)$

To find the particular integral

Since the partial differential equation is

$$F(D, D')\,z = f(x, y)$$

Therefore

$z = \dfrac{1}{F(D, D')} f(x, y)$ which gives the particular integral.

Note 1. $F(D, D') \dfrac{1}{F(D, D')} f(x, y) = f(x, y)$

Note 2. The symbolic function F(D, D') can be treated as an algebraic function of D and D' and can be factorized or expanded in ascending powers of D or D'. $\dfrac{1}{D}$ means integration w.r.t. x, $\dfrac{1}{D}$, means integration w.r.t. y, and so on and P.I. would be different if F(D, D') is expanded in ascending powers of D or D'.

Example 1. Find particular integral of

$(D^2+3DD'+2D'^2)\,z = x + y$

Solution. P.I. $= \dfrac{1}{(D^2 + 3DD' + 2D'^2)}(x+y)$

or $\quad z = \dfrac{1}{(D + 2D')(D + D')}(x + y)$

Let $u_1 = \frac{1}{(D+D')}(x+y)$ .... (*i*)

$\therefore \quad z = \frac{1}{(D+2D')}u_1$ .... (*ii*)

From (i) we have

$(D + D')u_1 = x + y$

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du_1}{x+y}$$

The first two relations give $dy-dx=0$ or $y-x=c_1$

Again from the relations $\frac{dx}{1} = \frac{du_1}{c_1+2x}$ $(\because y=c_1+x)$

We have $u_1 = c_1x + x^2$

$= (y-x)\,x + x^2$

$= xy$

With this value of u_1, we have from (*ii*)

$z = \frac{1}{(D+2D')}xy$ or $(D+2D')z = xy$

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{xy}$$

The first two relations give

$dy-2dx = 0$ or $y-2x = c_1$

Again from the relations

$$\frac{dx}{1} = \frac{dz}{x(c_1+2x)} \qquad (\because y = c_1+2x)$$

We have

$$z = c_1 \frac{x^2}{2} + 2\frac{x^3}{3}$$

$$=(y-2x)\frac{x^2}{2} + \frac{x^3}{3}$$

$$=\frac{1}{2}x^3 y - \frac{1}{3}x^3$$

Hence P.I. $= \frac{1}{2}x^2 y - \frac{1}{3}x^3$

1. Let $f(x, y)$ be a polynomial in x and y, then $f(x, y)=\Sigma A_{rs} x^r y^s$, where r, s are positive integers (or zero), and A_{rs} are constants.

In such a case we can use the method discussed above.

Another method will be to write.

F(D, D')=(D^2 + 3DD' + 2^{2}D')

$$=\frac{1}{D^2}\left[1+3\frac{D'}{D}+2\left(\frac{D'}{D}\right)^2\right]$$

then

$$\frac{1}{(D^2+3DD'+2D'^2)}(x+y)=\frac{1}{D^2\left[1+3\frac{D'}{D}+2\left(\frac{D'}{D}\right)\right]}(x+y)$$

$$=\frac{1}{D^2}\left[1+3\frac{D'}{D}+2\left(\frac{D'}{D}\right)^2\right]^{-1}(x+y)$$

$$=\frac{1}{D^2}\left[1-3\frac{D'}{D}\right](x+y)$$

$$= \frac{1}{D^2}\left[(x+y) - \frac{3}{D}(1)\right] \quad (\because\ D'(x+y) = 1)$$

$$= \frac{1}{D^2}\left[x + y - 3x\right]$$

$$= \frac{1}{D^2}\left[-2x + y\right]$$

$$= \frac{1}{D}(-x^2 + xy)$$

$$= -\frac{1}{3}x^3 + \frac{1}{2}x^2y$$

$$= \frac{1}{2}x^2y - \frac{1}{3}x^3$$

3. Let $f(x, y)$ be of the form $e^{ax + by}$

$$\therefore \qquad \frac{1}{F(D, D')}e^{ax+by} = \frac{1}{F(a,b)}e^{ax+by}$$

provided $F(a, b) \neq 0$

Example 2. Find particular integral of

$$(D^2 - DD' - 2D'^2)z = e^{3x+4y} + e^{2x+y}$$

Solution. We have P.I. for the first term

$$\text{P.I.} = \frac{1}{D^2 - DD' - 2^2D'},\ e^{3x+4y}$$

$$= \frac{1}{3^2 - 3.4 - 2.4^2}e^{3x+4y}$$

$$= -\frac{1}{35}e^{3x+4y}$$

For the second term $e^{2x + y}$

$$\text{P.I.} = \frac{1}{(D-2D')(D+D')} e^{2x+y} \qquad (\because\ F(a,\ b)=0)$$

$$= \frac{1}{(D-2D')}\left[\frac{1}{2+1} e^{2x+y}\right]$$

$$= \frac{1}{3}\frac{1}{D-2D'} e^{2x+y}$$

$$= \frac{1}{3} x e^{2x+y}$$

Let $u_1 = \frac{1}{(D-2D')} e^{2x+y}$ or $(D-2D')u_1 = e^{2x+y}$

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{du_1}{e^{2x+y}}$$

From the first of these two relations

$\frac{dx}{1} = \frac{dy}{-2}$ we have $y+2x=c_1$

Again from the relations $\frac{dx}{1} = \frac{du_1}{e^{c_1}}$ $(\because\ y+2x = c_1)$

We have $u_1 = xe^{c_1} = xe^{y+2x}$

3. Let $f(x,\ y)$ be of the form $\frac{\sin}{\cos}(ax+by)$

$$\therefore \quad \frac{1}{F(D^2, DD', D'^2)} \frac{\sin}{\cos}(ax+by)$$

$$=\frac{1}{F(-a^2,-ab,-b^2)}\frac{\sin}{\cos}(ax+by)$$

provided $F(-a^2, -ab, -b^2)\neq 0$

Example 3. Find particular integral of

$(D^2-DD'-2D^{2'})z = \sin(3x+2y)$

Solution. We have

$$\text{P.I.} = \frac{1}{D^2, DD^{'}, 2D^{12}}\frac{\sin}{\cos}(3x+2y)$$

$$=\frac{1}{-3^2-(-3.2)-2(-2^2)}\sin(3x+2y)$$

$$=\frac{1}{-9+6+8}\sin(3x+2y)$$

$$=\frac{1}{5}\sin(3x+2y)$$

Example 4. Find particular integral of

$(D^3 - 7DD^2D' - 6^3D')z = \cos(x-y)$

Solution. We have

$$\text{P.I.}=\frac{1}{(D^3-7D^2D'-6^3D')}\cos(x-y) \quad [\because D^2\cos(x-y)=-\cos(x-y)]$$

$$=\frac{1}{(D+D')(D^2-DD'-6^2D')}\cos(x-y) \qquad DD'\cos(x-y)=\cos(x-y)$$

$$=\frac{1}{(D+D')(-1-1+6)}\cos(x-y) \qquad D^{12}\cos(x-y)=-\cos(x-y)$$

$$=\frac{1}{4}\frac{1}{D+D'}\cos(x-y) \qquad (*)$$

Now let

$$u_1 = \frac{1}{(D+D')}\cos(x-y)$$

$\therefore$ (D+D') u_1=cos (x–y) which is in Lagrange's form.

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} - \frac{du_1}{\cos(x-y)}$$

From the first two relations

$$\frac{dx}{1} = \frac{dy}{1}$$

we obtain x–y=c

Further from the relations.

$$\frac{dx}{1} = \frac{du_1}{\cos c} \qquad (\because\ c=x-y)$$

we obtain $u_1 = x \cos c$

$= x \cos (x-y) \qquad (\because\ c=x-y)$

With this value of u_1, we have from (*)

$$\text{P.I.} = \frac{1}{4} x \cos (x-y)$$

A general method of finding the P.I.

Consider the equation.

(D–mD') $z = f(x, y)$

This can be written as

$p - mq = f(x, y)$

The auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(x,y)}$$

The first two relations give

$y + mx = c$

Further taking $\frac{dx}{1} = \frac{dz}{f(x,c-mx)}$ $[\because y = c - mx]$

we get $\quad z = \int f(x_1 c - mx)dx$

Thus $\quad z = \frac{1}{(D - mD')} f(x,y)$

$$= \int f(x,c-mx)dx$$

where the constant c is to be replaced by y+ in x after integration, as the particular integral is not to contain an arbitrary constant.

Now if the equation is F(D.D.') $z = f(x, y)$

where $F(D, D') = (D-m_1D')(D-m_2D')$

then $\quad$ P.I. $= \frac{1}{(D-m_1D')} \cdot \frac{1}{(D-m_2D')} f(x,y)$

This can now be evaluated by the repeated application of the above method.

Example 5 : Solve the partial differential equation

$$\frac{\sigma^2 z}{\sigma x^2} - \frac{\sigma^2 z}{\sigma z \sigma y} - 2\frac{\sigma^2 z}{\sigma y^2} = (2x^2+xy-y^2)\sin xy - \cos xy$$

Solution : The given equation can be written as

$(D^2-DD'-2^2D')Z$

$=(2x^2+xy-y^2)\sin xy-\cos xy$

For C.F.

The auxiliary equation is

$m^2-m-2=0$

or $\quad (m-2)(m+1)=0$

$\therefore \quad m=2, -1$

$\therefore \quad$ C.F. $= \phi_1(y+2x)+\phi_2(y-x)$

where ϕ_1 and ϕ_2 being arbitrary.

For P.I.

$$\text{P.I.} = \frac{1}{(D-2D')(D+D')}(2x^2+xy-y^2)\sin xy-\cos xy$$

$$=\frac{1}{(D-2D')}\int f(x, c+x)dx$$

$$=\frac{1}{(D-2D')}\int\Big[\Big\{2x^2+x(c+x)-(c+x)^2\Big\}$$

$$\sin x(c+x)-\cos x(c+x)\Big]dx$$

$$=\frac{1}{(D-2D')}\int\Big[(2x^2-cx-c^2)\sin(cx+x^2)$$

$$-\cos(cx+x^2)\Big]dx$$

$$=\frac{1}{(D-2D')}\Big[\int(x-c)(2x+c)\sin(cx+x^2)dx$$

$$-\int\cos(cx+x^2)dx\Big]$$

$$\frac{1}{(D-2D')}\left[\left\{(x-c)\frac{\cos(cx+x^2)}{-1}+\int\cos(cx+x^2)dx\right\}\right.$$

$$\left.-\int\cos(cx+x^2)dx\right]$$

integrating the first integral by parts

$$=\frac{1}{(D-2D')}(c-x)\cos(cx+x^2)$$

$$=\frac{1}{(D-2D')}(y-2x)\cos xy \qquad (\because\ y-x=c)$$

$= \int f(x, c-2x)dx$ $\quad (\because\ m=2)$

$= \int (c-4x) \cos x(c-2x)\ dx$

$= \sin x(c-2x)$

$= \sin xy$ $\quad (\because\ c-2x=y)$

$\therefore$ The complete solution is

$$z = \phi_1(y + 2x)+\phi_2(y - x) + \sin xy$$

Example 6. Solve the partial differential equation.

$$(D^2-D'^2)z = \tan^3 x \tan y - \tan x \tan^3 y$$

Solution. The given equation is

$$(D^2-D^{2'})z = \tan^3 x \tan y - \tan x \tan^3 y$$

For C.F.

The auxiliary equation is

$$m^2-1 = 0$$

or $m=1, -1$

$\therefore$ C.F. $= \phi_1(y+x)+\phi_2(y-x)$

where ϕ_1 and ϕ_2 are arbitrary.

For P.I.

$$\text{P.I.} = \frac{1}{(D-D')(D+D')} (\tan^3 x \tan y-\tan x \tan^3 y)$$

$$= \frac{1}{(D-D')}\frac{1}{D+D'} (\tan^3 x \tan y-\tan x \tan^3 y)$$

$$= \frac{1}{(D-D')}\int f(x, c-mx)dx$$

$$= \frac{1}{(D-D')}\int f(x, c+x)dx \quad (\because\ m=-1)$$

$$= \frac{1}{(D-D')}\int \left[\tan^3 x\ \tan(c+x) - \tan x \tan^3(c+x)\right] dx$$

$$=\frac{1}{(D-D')}\int \tan x \ \tan(c+x)[\tan^2 x-\tan^2(c+x)]dx$$

$$=\frac{1}{(D-D')}\int \tan x \ \tan(c+x)[\sec^2 x-\sec^2(c+x)]dx$$

$$=\frac{1}{(D-D')}\left[\int \tan x \ \tan(c+x)\sec^2 x dx-\int \tan x \ \tan(c+x)\sec^2(c+x)dx\right]$$

$$=\frac{1}{(D-D')}\tan(c+x)\frac{1}{2}\tan^2 x \int\frac{1}{2}\tan^2 x\sec^2(c+x)dx$$

$$-\tan x\frac{1}{2}\tan^2(c+x)+\int\frac{1}{2}\tan^2(c+x)\sec^2 x dx$$

$$=\frac{1}{2}\frac{1}{(D-D')}[\tan^2 x\tan(c+x)-\tan x\tan^2(c+x)$$

$$+\int\{\sec^2(c+x)-\sec^2 x\}dx]$$

$$=\frac{1}{2}\frac{1}{(D-D')}[\tan^2 x\tan(c+x)-\tan x\tan^2(c+x)+\tan(c+x)-\tan x]$$

$$=\frac{1}{2}\frac{1}{(D-D')}[\tan^2 x\tan y-\tan x\tan^2 y+\tan y-\tan x]$$

$$=\frac{1}{2}\frac{1}{(D-D')}(\tan y\sec^2 x-\tan x\sec^2 y) \qquad (\because c+x=y)$$

$$=\frac{1}{2}\int f(x,c-x)dx$$

$$=\frac{1}{2}\int[\tan(c-x)\sec^2 x-\tan x\sec^2(c-x)]\,dx$$

$$=\frac{1}{2}\left[\tan(c-x)\tan x+\int\tan x\sec^2(c-x)dx-\int\tan x\sec^2(c-x)dx\right]$$

$$=\frac{1}{2}\tan(c-x)\tan x$$

$= \frac{1}{2} \tan y \tan x \qquad (\because\ c - x = y)$

$\therefore$ the complete solution is

$$z = \phi_1(y+x) + \phi_2(y-x) + \frac{1}{2}\tan x \tan y$$

where ϕ_1 and ϕ_2 are arbitrary.

EXERCISE

Solve the following partial differential equations :

1. $(D + D')z = \sin x$
2. $(D^2 - 2DD' + D^{2'})\, z = e^{x + 2y}$
3. $(2D^2 - 5DD' + 2D^{2'})\, z = 24\,(y - x)$
4. $(D^2 - 5DD' + 4D^{2'})\, z = \sin\,(2x + 3y)$
5. $(D^2 - 2DD' + 4D^{2'})\, z = \sin\,(4x + y)$
6. $(D^2 - 6DD' + 9D^{2'})z = 6x + 2y$
7. $(D^2 - 2DD' + D^{2'})z = 2\cos y - x\sin y$
8. $(D^3 - DD^{2'} - DD^{2'} - D^{3'})\, z = e^y \cos 2x$
9. $(D^3 - 2DD^{2'} - DD^{2'} - 2^3D')\, z = (y + 2)\, e^x$
10. $(D^3 - 3DD^{2'} - 4DD^{2'} + 12D^{3'})\, z = \sin\,(2x+y)$

ANSWERS

1. $z = \phi\,(y-x) - \cos x$
2. $z = \phi_1(y + x) + x\,\phi_2(y + x) + e^{x + 2y}$
3. $z = \phi_1(y + 2x) + \phi_2(2y + x) + \frac{4}{9}(y - x)^3$
4. $z = \phi_1(y + x) + x\,\phi_2(y + x) - \sin\,(2x + 3y)$
5. $z = \phi_1(y + x) + \phi_2(y + 4x) - \frac{1}{3} x \cos\,(4x + y)$
6. $z = \phi_1(y + 3x) + x\,\phi_2(y + 3x) + x^2(3x + y)$
7. $z = \phi_1(y - x) + x\,\phi_2(y - x) + x \sin y$
8. $z = \phi_1(y + x) + \phi_2(y - x) + x\,\phi_3(y - x) - \frac{1}{25_e}^{y} \cos 2x - \frac{2}{25_e}^{y} \sin 2x$
9. $z = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y - 2x) + ye^x$
10. $z = \phi_1(y - 2x) + \phi_2(y + 2x) + \phi_3(y + 3x) \quad + \frac{1}{4} x \sin\,(2x + y)$

NON HOMOGENEOUS LINEAR PARTIAL DIFFERENTIAL EQUATIONS

A linear partial differential equation which is **not homogeneous** is called a **non-homogeneous** linear equation. Consider the partial differential equation.

$$F\ (D,\ D')\ z = f\ (x,\ y)$$

where F (D, D') is now not **necessarily** homogeneous. While F (D, D') when it is homogeneous, is always resolvable into linear factors, the same is not always true when F(D, D') is non-homogeneous. Therefore, we classify linear differential operators F(D, D') into two main types, which we shall treat separately. These are :

(i) F (D, D') is reducible if it can be expressed as product of linear factors of the form aD + bD' + c where a, b, c are constants.

(ii) F(D, D') is irreducible *i.e.*, when F(D, D') is not reducible for example $D^2 - D'$.

Complementary functions corresponding to linear factors.

We shall first consider the case when $aD+bD'+c$ be a **factor** of F(D, D'). To find C.F. corresponding to this factor, consider the most simple non-homogeneous equation. $(aD+bD'+c)z=0$

This can be written as $ap + bq + cz = 0$

The auxiliary system is

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{-cz}$$

and from the first two relations, we have on integration

$$ay - bx = A \qquad \text{.... } (i)$$

If $a \neq 0$ then from $\frac{dx}{a} = \frac{dz}{-cz}$

We have on integration

$$\log z = -\frac{c}{a} x + \log B$$

or $\quad z = B_e - \frac{c}{a} x$ .... (*ii*)

and the **general solution** may be expressed as

$$z = e^{-\frac{c}{a}x} \phi(ay - bx)$$

Further, if $b \neq 0$, then from $\frac{dy}{b} = \frac{dz}{-cz}$

we have on integration $z = C_e - \frac{cy}{b}$

and the general solution may be expressed as

$$z = e^{-\frac{c}{b}y} \Psi(ay - bx)$$

Note : If the linear factor is $D - mD'\gamma$, then the corresponding C.F. is $e^{\gamma x} \phi(y + mx)$

We now come to the **various cases** that arise :

(i) $F(D, D') = (a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2)$

i.e., when the factors are distinct, then C.F. or $F(D, D')\, z = 0$ is

$$z = e^{-\frac{c_1}{a_1}} \phi_1(a_1y - b_1x) + e^{-\frac{c_2}{c_1}x} \phi_2(a_2y - b_2x)$$

where ϕ_1 and ϕ_2 are arbitrary.

(ii) F(D, D') has repeated roots.

Let a factor aD + bD' + c occur twice in F(D, D')

Consider $(aD + bD' + c)^2 z = 0$ .... (1)

Take $(aD + bD' + c)\, z = 0$ .... (2)

∴ (1) becomes

$(aD + bD' + c)\, u = 0$ .... (3)

This gives $u = e^{-\frac{c}{a}x}\phi(ay - bx)$

With this value of u, we have from (2)

$$(aD + bD' + c)\, z = e^{-\frac{c}{a}x}\phi(ay - bx)$$

This equation can be written in the form

$$ap + bq = -cz + e^{-\frac{c}{a}x}\phi(ay - bx) \qquad \text{.... (4)}$$

which is of the form Pp + Qq = R.

The auxiliary equations are

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{-cz + e^{-\frac{c}{a}x}\phi(ay - bx)}$$

The first two relations give $ay - bx = A$

Again the relations

$$\frac{dx}{a} = \frac{dz}{-cz + e^{-\frac{c}{a}x}\phi(A)} \qquad (\because ay - bx = A)$$

give

$$\frac{dz}{dx} + \frac{c}{a}z = \frac{1}{a}e^{-\frac{c}{a}x}.\phi(A) \qquad \text{.... (5)}$$

which is a linear differential equation.

Its solution is

$$ze^{\frac{c}{a}x} = B + \frac{x}{a}\phi(A)$$

or $$z = e^{-\frac{c}{a}x}\,[\phi_1(ay - bx) + x\phi_2(ay - bx)] \quad \text{.... (6)}$$

where $B = \phi_1(A) = \phi_1(ay - bx)$

$$\frac{\phi(A)}{a} = \phi_1(A) = \phi_2(ay - bx)$$

Hence the general solution is given by (6)

Example 1. Solve the partial differential equation.

$(D + D' - 1)(D + 2D' - 2)\, z = 0$

Solution. The given partial differential equation is

$(D + D' - 1)(D + 2D' - 2)\, z = 0$

There being linear distinct factors.

C.F. $= e^x\, \phi_1(y - x) + e^{2x}\, \phi_2(y - 2x)$

where ϕ_1 and ϕ_2 are arbitrary.

Example 2. Solve the partial differential equation.

$(D - 2D' + 5)^2\, z = 0$

Solution. The given partial differential equation is

$(D - 2D' + 5)^2\, z = 0$

There are repeated linear factors.

$\therefore$ C.F. $= e^{-5x}\, [\phi_1(y + 2x) + x\, \phi_2(y + 2x)]$

where ϕ_1 and ϕ_2 are arbitrary.

The particular integral

The methods are similar to those for homogeneous equations. A few examples are solved below to illustrate the method.

Example 1. Solve the differential equation.

$(D^2 - DD' - 2D^{2'} + 2D + 2D')\, z$

$= e^{3x + 4y} + \sin(2x + 3y) + xy$

Solution. For C.F.

$$(D^2 - DD' - 2D'^2 + 2D + 2D')\ z = 0$$

or $[(D + D')(D - 2D') + 2(D + D')]\ z = 0$

or $(D + D')(D - 2D' + 2)\ z = 0$

$\therefore$ C.F. $= \phi_1(y - x) + e^{-2x}\ \phi_2(y + 2x)$

where ϕ_1 and ϕ_2 are arbitrary.

The portion of the P.I. for $e^{3x + 4y}$

$$= \frac{1}{(D^2 - DD' - 2D^{2'} + 2D + 2D')} e^{3x+4y}$$

$$= \frac{1}{3^2 - 3.4 - 2.4^2 + 2.3 + 2.4} e^{3x+4y}$$

$$= -\frac{1}{21} e^{3x+4y} \quad \text{for } \sin(2x + 3y)$$

$$= \frac{1}{(D^2 - DD' - 2^2 D' + 2D + 2D')} \sin(2x + 3y)$$

$$= \frac{1}{(-z^2) - (-2.3) - 2(-3)^2 + 2D + 2D'} \sin(2x + 3y)$$

$$= \frac{1}{2} \frac{1}{D + D' + 10} \sin(2x + 3y)$$

$$= \frac{1}{2} \frac{D + D' - 10}{(D + D')^2 - 100} \sin(2x + 3y)$$

$$= \frac{1}{2} \frac{D + D' - 10}{D^2 + 2DD' + D^2 - 100} \sin(2x + 3y)$$

$$=\frac{1}{2}\frac{2\cos(2x+3y)+3\cos(2x+3y)-10\sin(2x+3y)}{(-z^2)+2(-2\bullet 3)+(-3^2)-100}$$

$$=\frac{1}{250}[5\cos(2x+3y)-10\sin(2x+3y)]$$

$$=-\frac{1}{50}[\cos(2x+3y)-2\sin(2x+3y)]$$

for xy,

$$=\frac{1}{(D+D')(D-2D'+2)}xy$$

$$=\frac{1}{2}\frac{1}{D+D'}\left[1+\frac{(D-2D')}{2}\right]^{-1}xy$$

$$=\frac{1}{2}\frac{1}{(D+D')}\left[1-\frac{(D-2D')}{2}+\frac{(-1)(-2)}{21}\left(\frac{D-2D'}{2}\right)^2+........\right]xy$$

$$=\frac{1}{2}\frac{1}{(D+D')}\left[1-\frac{1}{2}D+D'-DD'\right]xy \quad \left(\because D^2(xy)=0^2D'(xy)=0\right)$$

$$=\frac{1}{2}\frac{1}{(D+D')}\left[xy-\frac{1}{2}y+x-1\right]$$

$$=\frac{1}{2}\frac{1}{D}\left[1+\frac{D'}{D}\right]^{-1}\left(xy-\frac{1}{2}y+x-1\right)$$

$$=\frac{1}{2}\frac{1}{D}\left[1-\frac{D'}{D}\right]\left(xy-\frac{1}{2}y+x-1\right)$$

$$=\frac{1}{2}\frac{1}{D}\left[xy-\frac{1}{2}y+x-1-\frac{1}{D}\left(x-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\frac{1}{D}\left[xy - \frac{1}{2}y + x - 1 - \frac{x^2}{2} + \frac{x}{2}\right]$$

$$= \frac{1}{2}\frac{1}{D}\left[xy - \frac{1}{2}y + \frac{3x}{2} - 1 - \frac{x^2}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2}y - \frac{1}{2}xy + \frac{3}{4}x^2 - x - \frac{x^3}{6}\right]$$

$$= \frac{1}{24}\ [6x^2y–6xy+9x^2–12x–2x^3]$$

$$= \frac{x}{24}\ [6xy–6y+9x–2x^2–12]$$

Hence the general solution is

$$z = \phi_1\ (y - x) + e^{-2x}\ \phi_2\ (y + 2x) - \frac{1}{21}\ e^{3x+4y}$$

$$-\frac{1}{50}[\cos\ (2x+3y)–2\ \sin\ (2x+3y)] + \frac{x}{24}\ [6xy–4y+6x–2x^2–12]$$

Example 2. Solve the differential equation.

$(D^2 - DD' - 2D^{12} + 6D–9D' + 5)\ z = e^{x+y} + (x^2 + 2y)\ e^{2x+y}$

Solution. For C. F. we have

$(D^2 - DD' - 2\ D^{2'} + 6D - 9D' + 5)\ z = 0$ (1)

Note : We write $D^2 - DD' - 2D^{2'} + 6D - 9D' + 5 = 0$

in the form $D^2 - (D' - 6) + (–2D^{2'} - 9D' + 5) = 0$

which is quadratic in D.

Solving, we have

$$D = \frac{(D'-6) \pm \sqrt{(D'-6)^2 + 8D^{2'} + 36\ D' - 20}}{2}$$

$$= \frac{(D'-6) \pm \sqrt{9D'+24\,D'+16}}{2}$$

$$= \frac{(D'-6) \pm (3D'+4)}{2}$$

$$= \frac{4D'-2}{2}, \frac{-2D'-10}{2}$$

$=2D' - 1, -D' - 5$

$\therefore$ the expression $D^2 - DD' - 2D^{2\prime} + 6D - 9D' + 5$ is equivalent to the product of the factors $(D - 2D' + 1)$ and $D + D' + 5$.

With this we can write (1) as

$(D - 2D' + 1)(D + D' + 5)\,z = 0$

whose solution is

$z = e^{-x}\,\phi_1\,(y + 2x) + e^{-5x}\,\phi_2\,(y - x)$

where ϕ_1 and ϕ_2 are arbitrary.

The portion of P. I.

for $e^{x+y} = \dfrac{1}{(D-2D'+1)(D+D'+5)} e^{x+y}$

$$= \frac{1}{(D-2D'+1)(D+D'+5)} e^{x+y}$$

$$= \frac{1}{(D-2D'+1)} \frac{1}{7} e^{x+y}$$

$$= \frac{1}{7} \frac{1}{(D-2D'+1)} e^{x+y} \quad \text{(case of failure)}$$

$$= \frac{1}{7} e^{x+y} \frac{1}{(D+1)-2(D'+1)+1}$$

$$\left(\because \frac{1}{F(D,D')} e^{ax+by} V = e^{ax+by} \frac{1}{F(D+a, D+b)}\right)$$

$$= \frac{1}{7} e^{x+y} \frac{1}{(D-2D')} 1$$

$$= \frac{1}{7} e^{x+y} \frac{1}{D\left[1-2\frac{D'}{D}\right]} 1$$

$$= \frac{1}{7} e^{x+y} \frac{1}{D}\left[1-2\frac{D'}{D}\right]^{-1}$$

$$= \frac{1}{7} e^{x+y} \frac{1}{D}\left[1+2\frac{D'}{D}\right] 1$$

$$= \frac{1}{7} e^{x+y} \frac{1}{D} \; 1$$

$$= \frac{x}{7} e^{x+y}$$

for $(x^2 + 2y)\ e^{2x+y}$

$$= \frac{1}{(D-2D'+1)(D+D'+5)} (x^2+2y)\ e^{2x+y}$$

$$= e^{2x+y} \frac{1}{(D+2)-2(D'+1)+1} \quad \frac{1}{(D+2)-2(D'+1)+5} (x^2+2y)$$

$$= e^{2x+y} \frac{1}{(D-2D'+1))} \frac{1}{(D+D'+8)} (x^2+2y)$$

$$= e^{2x+y} \frac{1}{(D-2D'+1)}\left[1+\frac{D+D'}{8}\right]^{-1} (x^2+2y)$$

$$= \frac{1}{8} e^{2x+y} \frac{1}{(D-2D'+1)}\left[1-\frac{1}{8}D-\frac{1}{8}D'+\frac{1}{64}D^2\right]^{-1}(x^2+2y)$$

$$= \frac{1}{8} e^{2x+y} \frac{1}{(D-2D'+1)}\left[x^2+2y-\frac{1}{4}x-\frac{1}{4}+\frac{1}{32}\right]$$

$$= \frac{1}{8} e^{2x+y} \frac{1}{(D-2D'+1)}\left[x^2-\frac{1}{4}x+2y-\frac{7}{32}\right]$$

$$= \frac{1}{8} e^{2x+y} \left[1+(D-2D')\right]^{-1}\left(x^2-\frac{1}{4}x+2y-\frac{7}{32}\right)$$

$$= \frac{1}{8} e^{2x+y} \left[1-(D-2D')+D^2\right]\left(x^2-\frac{1}{4}x+2y-\frac{7}{23}\right)$$

$$= \frac{1}{8} e^{2x+y} \left[1+2D'-D+D^2\right]\left(x^2-\frac{1}{4}x+2y-\frac{7}{32}\right)$$

$$= \frac{1}{8} e^{2x+y} \left[\begin{array}{ll} x^2-\frac{1}{4}x+2y-\frac{7}{32} & \\ \quad -2x & +4 \\ & +\frac{1}{4} \\ & +2 \end{array}\right]$$

$$= \frac{1}{8} e^{2x+y} \left[x^2-\frac{9}{4}x+2y+\frac{193}{32}\right]$$

$$= \frac{1}{256} e^{2x+y} \; [32x^2 - 72x+64y = 193]$$

Hence the general solution is

$$z = e^{-x} \phi_1 (y + 2x) +| e^{-5x} \phi_2 (y - x) + \frac{1}{7} xe^{x+y}$$

$$+\frac{1}{256} e^{2x+y} (32x^2 - 72x + 64y + 193)$$

where ϕ_1 and ϕ_2 are arbitrary.

Irreducible equations. Complementary Function.

Let us consider the equation with constant coefficients.

$$F(D, D')\, z = 0 \qquad (1)$$

Since $D^r D'^s (c\, e^{ax+by}) = c\, a^r b^s e^{ax+by}$

where a, b, c are constants, the result of subtituting $z = c\, e^{ax+by}$ in the given equation is $cF(a, b)\, e^{ax+by} = 0$

Thus $z = c\, e^{ax+by}$ is a solution of the equation (1) provided $F(a, b) = 0$, with arbitrary.

Now for any value of a (or b), one or more values of b (or a) are obtained from the relation $F(a, b) = 0$. Thus, there exist **infinitely many** pairs of numbers (a_r, b_r) satisfying the relation, and so

$$z = \sum_{r=1}^{\infty} c_r\, e^{a_r x + b_r y}, \; F(a_r, b_r) = 0$$

is a solution of the given equation *i.e,* the complementary function of the equation.

$$F(D, D')\, z = f(x, y)$$

Example 1. Solve the differential equation.

$$(D^2 + D + D')\, z = 0$$

Solution. We have

$$F(D, D') = D^2 + D + D'$$

$\therefore$ $F(a, b) = 0$ gives $a^2 + a + b = 0$

so that for any $a = a_r$, $b_r = -(a_r + 1)\, a_r$

Thus the solution is

$$z = \sum_{r=1}^{\infty} c_r\, e^{a_r x - a_r (a_r + 1) y}$$

$r = 1$

with c_r and a_r arbitrary constants.

Example 2. Solve the differential equation.

$$(D + 2D')(D - 2D' + 1)(D - D^{2\prime})\, z = 0$$

Solution : Corresponding to the linear factors, we have $\phi_1\,(y - 2x)$ and $e^x\,\phi_2\,(y + 2x)$ respectively. For the irreducible factor $D - D^{2\prime}$ we have $a - b^2 = 0$ *i.e,* $a = b^2$.

Hence the solution is

$$z = \phi_1\,(y - 2x) + e^x\,\phi_2\,(y + 2x) + \sum_{r=1}^{\infty} c_r\, e^{b_r^2 x + b_r y}$$

with c_r and b_r arbitrary constants.

EXERCISE

Solve the following partial differential equations.

1. $(2D + 3D' - 5)(D + 2D')(D-2)(D'+2)\, z = 0$
2. $(2D + D' + 5)(D - 2D' + 1)^2\, z = 0$
3. $D\,(2D - D' + 1)(D + 2D' - 1)\, z = 0$
4. $(2DD' + 2D^{2\prime} - 3D')\, z = 3\cos(3x - 2y)$
5. $D\,(D + D' - 1)(D + 3D' - 2)\, z = x^2 - 4xy + 2y^2$
6. $(D^2 - D^{2\prime} - 3D + 3D')\, z = xy + e^{x+2y}$
7. $(D^2 - D^{2\prime} + D + 3D' - 2)\, z = e^{x-y} - x^2 y$
8. $(D^2 + DD' - D^2 + D - D')\, z = e^{2x-3y}$
9. $(D^2 + 2DD^{2\prime} - 2D' + 3)\, z = e^{x+y}\cos(x+2y)$

ANSWERS

1. $z = e^{\frac{5x}{2}}\,\phi_1\,(2y-3x) + \phi_2\,(y-2x) + e^{2x}\phi_3(y) + e^{-2y}\phi_4(x)$

2. $z = e^{-5y}\phi_1(2y-x) + e^{-x}[\phi_2(y + 2x) + x\,\phi_3(y + 2x)]$

3. $z = \phi_1(y) + e^{y}\phi_2(2y + x) + e^{x}\phi_3(y-2x)$

4. $z = \phi_1(x) + e^{3y}\phi_2(2y-x) + \frac{3}{50}$ [4 cos $(3x-2y)$

+ 3 sin $(3x-2y)$

5. $z = \phi_1(y) + e^{x}\phi_2(y-x) + e^{2x}\phi_3(y - 3x)$

$+ \frac{1}{12}(2x^3-12x^2y+12xy^2-21x^2+24xy+3x)$

6. $z = \phi_1(x + y) + e^{3y}\phi_2(y-x)$

$-\left(ye^{x+2y} + \frac{1}{18}x^3 - \frac{1}{2}x^2y + \frac{1}{9}x^2 + \frac{1}{9}xy + \frac{2}{27}x\right)$

7. $z = e^{x}\phi_1(y-x) + e^{-2x}\phi_2(y+x)\frac{1}{4}e^{x-y} + \frac{1}{2}x^2y$

$+\frac{3}{4}x^2 + \frac{1}{2}xy + \frac{3}{2}x = \frac{3}{4}y + \frac{21}{8}$

8. $z = \sum_{r=1}^{\infty} c_r e^{a_r x + b_r y} - \frac{1}{6}e^{2x-3y}$

where $a^2_r + a_r b_r - b^2_r + a_r - b_r = 0$

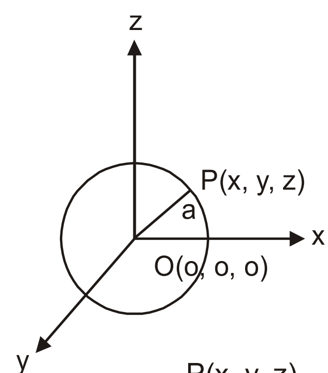
9. $z = \sum_{r=1}^{\infty} c_r e^{a_r x + b_r y} - \frac{1}{13}e^{x+y}\cos(x + 2y)$

where $a_r^2 + 2a_r b_r - 2b_r + 3 = 0$

4.1.0 SPHERE

It is the locus of a point which moves such that its distance from a fixed point always remains contants. The fixed point is called the centre of the sphere and the constant distance is called the radius of the sphere.

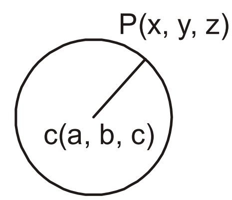
4.1.1 THE EQUATION OF A SPHERE IN STANDARD FORM



The equation of the sphere whose centre is origin and radius 'a' is given by

$$x^2 + y^2 + z^2 = a^2$$

4.1.2 EQUATION OF A SPHERE WITH GIVEN CENTRE AND RADIUS



The equation of a sphere whose centre is (a, b, c) and the radius r is given by $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$

4.1.3 EQUATION OF A SPHERE IN GENERAL FORM

To prove that the equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere. Find its centre and radius.

Proof : The given equation is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad ...... (1)$$

$(x^2 + 2ux) + (y^2 + 2uy) + (z^2 + 2wz) = -d$ ...... (2)

Adding $u^2 + v^2 + w^2$ to both sides of eq (2) to complete the perfect square, we get.

$$\left(x^2 + 2ux + u^2\right) + \left(y^2 + 2vy + v^2\right) + \left(z^2 + 2wz + w^2\right) = u^2 + v^2 + w^2 - d$$

$$\Rightarrow [x - (-u)]^2 + [y - (-v)]^2 + [z - (-w)]^2 = \left(\sqrt{u^2 + v^2 + w^2 - d}\right)^2 \quad \text{...... (3)}$$

which is clearly equation of sphere in central form

i.e., $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ ...... (4)

Compairing (3) and (4), we get

$a = -u$, $b = -v$, $c = -w$ and $r = \sqrt{u^2 + v^2 + w^2 - d}$

Hence centre of the sphere (1) is

$(-u, -v, -w)$ and its centre is $\sqrt{u^2 + v^2 + w^2 - d}$

Note : The given equation represents a sphere if it satisfy the following conditions :-

i) It is a second degree equation in x, y, z.

ii) Coefficient of x^2 = coefficient of y^2 = coefficient of z^2, and

iii) It doesn't contain the term involving the products xy, yz, and zx.

Example : Find the equation of the sphere whose centre is (2, -3, 4) and radius 5.

Solution : Equation of the sphere with centre (2, -3, -4) and radius 5 is

$(x - 2)^2 + (y - (-3))^2 + (z - 4)^2 = (5)^2$

$\Rightarrow$ $x^2 - 4x + 4 + y^2 + 6y + 9 + z^2 - 8z + 16 = 25$

$x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$

Example: Find the centre and radius of the sphere given by

i) $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$

ii) $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z - 15 = 0$

Solution : i) The equation of the sphere is

$$x^2 + y^2 + z^2 - 2x + 4y - 6z - 11 = 0 \qquad \text{...... (1)}$$

Compare eq (1) with general equation of the sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get

$2u = -2,\ 2v = 4,\ 2w = -6,\ d = -11$

or $u = -1,\ v = 2,\ w = -3,\ d = -11$

Hence the given equation represents a sphere whose centre is

$(-u, -v, -w)$ i.e., $(1, -2, 3)$ and radious = $\sqrt{u^2 + v^2 + w^2 - d}$

i.e., $\sqrt{1+4+9+11} = \sqrt{25} = 5$

ii) The equation of the sphere is

$2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z - 15 = 0$

$$\Rightarrow \quad x^2 + y^2 + z^2 - x + 2y + z - \frac{15}{2} = 0 \qquad \text{...... (1)}$$

Compare eq (1) with the general equation of the sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get

$2u = -1,\ 2v = 2,\ 2w = 1,\ d = -\frac{15}{2}$

or $u = -\frac{1}{2}, v = 1, w = \frac{1}{2}, d = -\frac{15}{2}$

Hence the given equation represents a sphere whose centre is

(-u, -v, -w), i.e. $\left(\frac{1}{2}, -1, -\frac{1}{2}\right)$

and radius = $\sqrt{u^2 + v^2 + w^2 - d}$ = $\sqrt{\left(\frac{1}{2}\right)^2 + 1 + \left(\frac{1}{2}\right)^2 + \frac{15}{2}}$

= $\sqrt{\frac{1}{4} + \frac{1}{4} + 1 + \frac{15}{2}}$ = $\sqrt{\frac{1}{2} + \frac{15}{2} + 1}$

= $\sqrt{8+1}$ = $\sqrt{9}$ = 3

Example : Find the equation to the sphere whose centre is the point (-1, 2, 3) and which passes through the point (1, -1, 2)

Solution : Equation of the sphere having centre (-1, 2, 3) and radius 'a' is

$(x - (-1))^2 + (y - 2)^2 + (z - 3)^2 = a^2$

$\Rightarrow$ $(x + 1)^2 + (y - 2)^2 + (z - 3)^2 = a^2$ ...... (1)

Since it passes through (1, -1, 2), then we have

$(1 + 1)^2 + (-1 - 2)^2 + (2 - 3)^2 = a^2$

$\Rightarrow$ $4 + 9 + 1 = a^2$

$\Rightarrow$ $a^2 = 14$

With radius $a = \sqrt{14}$, the equation of the sphere (1) becomes

$(x + 1)^2 + (y - 2)^2 + (z - 3)^2 = 14$

$x^2 + 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9 = 14$

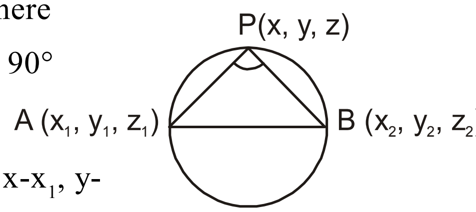
$\Rightarrow$ $x^2 + y^2 + z^2 + 2x - 4y - 6z = 0$

4.1.4 EQUATION OF A SPHERE IN DIAMETER FORM

To find the equation of the sphere whose ends of diameter are (x_1, y_1, z_1) and (x_2, y_2, z_2)

Proof : Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be the given end points of the diameter AB on the sphere. Let (P (x, y, z) be any point on the sphere. Join AP and BP.

Since AB is the diameter of the sphere



$\Rightarrow$ $\angle APB$ = angle in the semi circle = 90°

$\Rightarrow$ $AP \perp PB$

Now the direction ratio's of AP are $x-x_1$, $y-y_1$, $z-z_1$ and the direction ratios of BP are $x-x_2$, $y-y_2$, $z-z_2$.

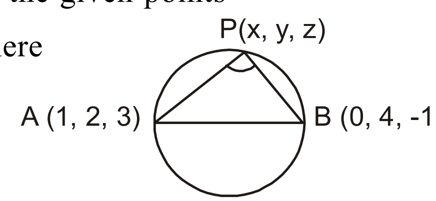
Since $AP \perp PB$

$\Rightarrow$ $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

Example : Obtain the equation of the sphere described on the join of (1, 2, 3) and (0, 4, -1) as diameter.

Solution : Let A (1, 2, 3) and B (0, 4, -1) be the given points

Let P (x, y, z) be any point on the sphere



Join PA and PB, then $\angle APB = 90°$

$\therefore$ $AP \perp PB$ ...... (1)

Now the direction ratios of AP are x - 1, y - 2, z - 1 and those of BP are x - 0, y - 4, z + 1

$\therefore$ From eq (1), we get

$(x - 1)(x - 0) + (y - 2)(y - 4) + (z - 3)(z + 1) = 0$

$x^2 - x + y^2 - 4y - 2y + 8 + 4z^2 + z - 3z - 3 = 0$

$\Rightarrow$ $x^2 + y^2 + z^2 - x - 6y - 2z + 5 = 0$

which is the required equation of the sphere.

4.1.5 EQUATION OF A SPHERE IN FOUR POINTS FORM

To find the equation of a sphere passing through four given points.

Proof : Let the required equation of sphere be

$x^2 + y^2 + z^2 - 2ux + 2vy + 2wz + d = 0$ ...... (1)

Let P (x_1, y_1, z_1), Q (x_2, y_2, z_2), R (x_3, y_3, z_3) and S (x_4, y_4, z_4) be the four given points on the sphere

Since (1) passes through P, Q, R, and S, we have

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0 \quad ...... (2)$$

$$x_2^2 + y_2^2 + z_2^2 + 2ux_2 + 2vy_2 + 2wz_2 + d = 0 \quad ...... (3)$$

$$x_3^2 + y_3^2 + z_3^2 + 2ux_3 + 2vy_3 + 2wz_3 + d = 0 \quad ...... (4)$$

$$x_4^2 + y_4^2 + z_4^2 + 2ux_4 + 2vy_4 + 2wz_4 + d = 0 \quad ...... (5)$$

Eliminating u, v, w, d from (1), (2), (3), (4) and (5) with the help of determinants, we get

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

which is the regd equation of the sphere in four point form

Note : In numerical examples, find the values of u, v, w, and d from (2), (3), (4) and (5) by substituting these values u, v, w, and d, in eq (1) we get the equation of sphere passing through four given points.

Example : Find the equation of the sphere through the points

(0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c)

Solution : Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ...... (1)

Since it passed through (0, 0, 0) $\Rightarrow$ d =0 ...... (2)

Again (1) passes through the points (a, 0, 0), (0, b, 0), (0, 0, c)

$\therefore$ $a^2 + 2ua + d = 0, b^2 + 2vb + d = 0, c^2 + 2wz + d = 0$

Putting d = 0 from (2), in these equations, we get

$a^2 + 2ua = 0, b^2 + 2vb = 0, c^2 + 2wc = 0$

$\Rightarrow$ $a(a + 2u) = 0, b(b + 2v) = 0, c(c + 2w) = 0$

$\Rightarrow$ $u = -\frac{a}{2}, v = -\frac{b}{2}, w = -\frac{c}{2}$

Substituting these values, u, v, w, d in eq. (1), we get

$x^2 + y^2 + z^2 - ax - by - cz = 0$

EXERCISE

1. Find the equation of the sphere whose centre is (2, -3, 4) and radius 5.
2. Find the centre and radius of the spheres.
   i) $x^2 + y^2 + z^2 + 2x - 4y - 6z + 5 = 0$
   ii) $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 3 = 0$
3. Prove that the equation $ax^2 + ay^2 + az^2 + 2ux + 4uy + 2wz + d = 0$ represents a sphere and find its centre and radius.
4. Find the equation of the sphere if (-1, 3, 2) (5, 7, -6) are the end of a diameter.
5. Find the equation of a sphere which passes through (0, 0, 0) and which has its centre at $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$

6. Find the equation of the sphere through the following points and also find its centre and radius.

(0, 0, 0), (-1, 1, 1), (1, -1, 1) and (1, 1, -1)

7. Find the equation of the sphere circumscribing the tetrahedron

$$x = 0,\ y = 0,\ z = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

8. Find the equation of sphere which passes through the points (1, -4, 3), (1, -5, 2), (1, -3, 0) and whose centre lies on the plane x + y + z =0

9. Show that the equation of the sphere passing through the three points (3, 0, 2) (-1, 1, 1), (2, -5, 4) and having its centre on the plane $2x + 3y + 4z = 6$ is $x^2 + y^2 + z^2 + 4y - 6z - 1 = 0$

ANSWERS

1. $x^2 + y^2 + z^2 - 4x + 6y - 8z + 24 = 0$

2. (i) c (-1, 2, 3), radius = 3 (ii) $c\left(\frac{1}{2}, -1, \frac{-1}{2}\right)$, radius = 0

3. $\left(\frac{-u}{a}, \frac{-v}{a}, \frac{-w}{a}\right)$; $\frac{1}{a}\sqrt{u^2 + v^2 + w^2 - d}$

4. $x^2 + y^2 + z^2 - 4x - 10y + 4z + 4 = 0$

5. $2(x^2 + y^2 + z^2) = 1$

6. $x^2 + y^2 + z^2 - 3x - 3y - 3z = 0$; $\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)$; $\frac{3\sqrt{5}}{2}$

7. $x^2 + y^2 + z^2 - ax - by - cz = 0$

8. $x^2 + y^2 + z^2 - 4x + 7y - 3z + 15 = 0$

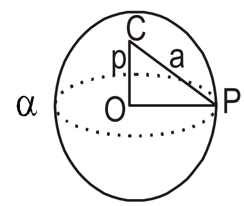
****

4.2.1 PLANE SECTION OF A SPHERE

To prove that the section of a sphere by a plane is a circle.

Proof : Let C be the centre of the sphere, 'a' is its radius and α the plane. Draw $CO \perp$ from C on the plane Land let CO = p, so that O is a fixed point and p is fixed length.

Let P be any point on the section of the sphere by the plane α. Join CP and OP. Since CO to the plane α.



$CO \perp OP$, a line which means it is that plane.

$\therefore$ In the right $\angle d$ ΔCOP, we have

$OP^2 = (CP)^2 - (CO)^2 = a^2 - p^2$

or $OP = \sqrt{a^2 - p^2}$

which is constant and O is a fixed point.

Hence p lies on a circle whose centre is O and radius $\sqrt{a^2 - p^2}$. This proves that the section of a sphere by a plane is a circle.

Note: 1) The centre of circle is the foot of perpendicular from the centre of the sphere on the plane, and

2) Radius of the circle = $\sqrt{a^2 - p^2}$ where 'a' is the radius of sphere and 'p', the length of $\perp$ from the centre of the sphere on the plane.

3) If p = 0, then circle is called great circle.

4.2.2 THE EQUATION TO A CIRCLE

Any circle is the intersection of a plane and a sphere, hence a circle can be represented by two equations, one being of a sphere and the other of plane. Thus the equations

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$lx + my + nz = p$

taken together represent a circle.

4.2.3 INTERSECTION OF TWO SPHERES

To show that the curve of intersection of two spheres is a circle.

Proof : Let the two spheres be given by the equations.

$S_1 : x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$ ...... (1)

and $S_2 : x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$ ...... (2)

Then the coordinates of points common to both the spheres satisfy both the equations and hence they satisfy the equation.

$S_1 - S_2 = 2(u_1 - u_2) + 2(v_1 - v_2) + 2(w_1 - w_2)z + d_1 - d_2 = 0$ ...... (3)

which being first degree equation in x, y, z represent a plane.

Thus, the curve of intersection of two spheres is the same as that of the plane (3) with any of the given spheres.

Example : Find the centre and radius of the circle in which the sphere

$x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$

is cut by the plane $x + 2y + 2z + 7 = 0$

Solution : The given sphere is

$x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$ ...... (1)

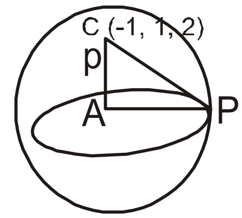
Its centre is c(-1, 1, 2) and radius $\sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{1+1+4+19} = 5$

The given plane is $x + 2y + 2z + 7 = 0$ ...... (2)

Equation (1) and (2) taken together represent a circle. Now the centre of the circle is the foot of $\perp$ from the centre of the sphere (1) on the plane (2)

Now direction ratio's of the normal to the plane (2) are 1, 2, 2 equation of the liine CA through C and perpendicular to plane (2) are



$$\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2} = r \text{ (say)}$$

At any point on this line is A (r - 1, 2r + 1, 2r + 2) ...... (3)

Since A lies on the plane (2)

$\therefore$ $(r-1) + 2(2r + 1) + 2(2r + 2) + 7 = 0$

$\Rightarrow$ $r - 1 + 4r + 2 + 4r + 4 + 7 = 0$

$\Rightarrow$ $9r + 12 = 0$

$\Rightarrow$ $r = -\frac{4}{3}$

Putting this value of r in eq (3), we get

$A\left(-\frac{7}{3}, -\frac{5}{3}, -\frac{2}{3}\right)$, which is the required centre of the circle.

Again p = CA = $\perp$ distance from C (-1, 1, 2) on the sphere (2)

$$p = \frac{-1+2(1)+2(2)+7}{\sqrt{1^2+2^2+2^2}}$$

$$= \frac{-1+2+4+7}{3} \qquad = \frac{12}{3} \quad = 4$$

Also CP = radius of the sphere a = 5

$\therefore$ Radius of the circle $= AP = \sqrt{a^2 - p^2} = \sqrt{CP^2 - CA^2}$

$$= \sqrt{25-16} = 3$$

Example : Prove that the equation of the sphere which passes through the point (α, β, γ) and the circle $x^2 + y^2 + z^2 = a^2$, $z = 0$ is

$$\gamma\,(x^2 + y^2 + z^2 - a^2) = z\,(\alpha^2 + \beta^2 + \gamma^2 - a^2)$$

Solution : The equation of the sphere through the given circle is

$$x^2 + y^2 + z^2 - a^2 + \lambda z = 0 \qquad \text{...... (1)}$$

It will pass through the point (α, β, λ)

if $\alpha^2 + \beta^2 + \gamma^2 - a^2 + \lambda\gamma = 0$

$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 - a^2 = -\lambda\gamma$

$\Rightarrow \quad \lambda = \dfrac{(\alpha^2 + \beta^2 + \gamma^2 - a^2)}{\gamma}$

Substituting the value of λ in equation (1), we get

$$\gamma\,(x^2 + y^2 + z^2 - a^2) = z\,(\alpha^2 + \beta^2 + \gamma^2 - a^2)$$

EXERCISES

1. Find the centre and radius of the circle

   $x^2 + y^2 + z^2 + 25, 2x + y +2z = 9.$

2. Show that the radius of the circle

   $x^2 + y^2 + z^2 + x + y + z = 4 = 0, x + y + z = 0$ is 2.

3. Obtain the equations of the circle lying on the sphere

   $x^2 + y^2 + z^2 - 2x + 4y - 6z + 3 = 0$ and having its centre at (2, 3, -4)

4. Find the centre and radius of the circle

   $x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0, x -2y + 2z = 3.$

4.2.4 SPHERES THROUGH A GIVEN CIRCLE

Let the circle be given by the equations

$S : x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ .... (1)

$P : lx + my + nz - p = 0$ .... (2)

Now consider the equations $S + kP = 0$

$\Rightarrow$ $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d + k(lx + my + nz - p) = 0$ ..... (3)

Note: The above equations (3) represents a sphere through the curve of intersection of (1) and (2) in the given circle. Thus any sphere through the circle $S = 0$, $P = 0$ is $S + kP = 0$

Similarly, if the circle be given by the intersecton of two spheres :

$S : x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$S^1 : x^2 + y^2 + z^2 + 2u^1x + 2v^1y + 2w^1z + d^1 = 0$

Then any sphere through this circle is $S + k\,S^1 = 0$

Also, the equation of the planes of the circle through the two spheres

$S = 0$, $S^1 = 0$ is

$S - S^1 = 2(u - u^1)x + 2(v - v^1) + 2(w - w^1)z + d - d^1 = 0$

From this, we see that the equation of any sphere through the circle $S = 0$, $S^1 = 0$ may also be taken as

$S + k(S - S^1) = 0$

Definition : The section of a sphere by a plane passing through its centre is called a great circle.

The centre and radius of the great circle are the same as the centre and radius of the sphere.

Note : If a plane meets a sphere in a great circle, then the centre of the sphere lies on the plane of the circle.

Example : Obtain the equation of the sphere through the circle

$x^2 + y^2 + z^2 = 9,\ 2x + 3y + 4z = 5$ and the origin

Solution : Any sphere through the circle

$x^2 + y^2 + z^2 = 9,\ 2x + 3y + 4z = 5$ is

$x^2 + y^2 + z^2 - 9 + k(2x + 3y + 4z - 5) = 0$ .... (1)

Since it passes through the origin the (0, 0, 0)

$\therefore \quad -9 - 5k = 0 \quad \Rightarrow -9 = 5k \quad \Rightarrow k = \frac{-9}{5}$

Substituting the value of k in equation (1), we get

$x^2 + y^2 + z^2 - 9\ \frac{-9}{5}\ (2x + 3y + 4z - 5) = 0$

$\Rightarrow \quad 5(x^2 + y^2 + z^2) - 45 - 18x - 27y - 36z + 45 = 0$

$\Rightarrow \quad 5(x^2 + y^2 + z^2) - 18x - 27y - 36z = 0$

Example : A circle with centre (2, 3, 0) and radius 1 is drawn in the plane z = 0. Find the equation of the sphere which passes through this circle at the point (1, 1, 1)

Solution : Equation of the given circle is

$(x - 2)^2 + (y - 3)^2 + z^2 = 1^2,\ z = 0$ .... (1)

Any sphere through the above circle is

$(x - 2)^2 + (y - 3)^2 + z^2 - 1^2 + k = 0$ .... (2)

where k is constant

If it passes through (1, 1, 1), then

$(1 - 2)^2 + (1-3)^2 + (1)^2 - 1^2 + k = 0$

$\Rightarrow \quad 1 + 4 + k = 0 \quad \Rightarrow \quad k = -5$

Substitute the value of k in eq (2), we get

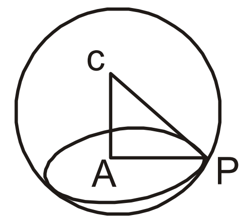
$$(x - 2)^2 + (y - 3)^2 + z^2 - 1 - 5z = 0$$

$\Rightarrow \quad x^2 - 4x + 4 + y^2 - 6y + 9 + z^2 - 5z - 1 = 0$

$\Rightarrow \quad x^2 + y^2 + z^2 - 4x - 6y - 5z + 12 = 0$

Example : Prove that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x - 2 = 0$ in the circle of radius unity; and find the equation of the sphere which has this circle for one of the great circle.

Solution : The given sphere is



$x^2 + y^2 + z^2 - x + z - 2 = 0$ .... (1)

and the plane $x + 2y - z - 4 = 0$ .... (2)

Now the centre of the sphere (1) is $C\left(\frac{1}{2}, 0, \frac{-1}{2}\right)$

and its radius $\sqrt{\left(\frac{1}{2}\right)^2 + 0^2 + \left(\frac{-1}{2}\right)^2 + 2}$

$$= \sqrt{\frac{5}{2}}$$

$\therefore \quad CA = \perp$ distance of $c\left(\frac{1}{2}, 0, \frac{-1}{2}\right)$ from the plane (2)

$$= \frac{\frac{1}{2} + 2(0) - \left(\frac{-1}{2}\right) - 4}{\sqrt{1+4+1}} = \frac{3}{\sqrt{6}}$$

$$= \frac{3}{\sqrt{2}\sqrt{3}} \text{ x } \sqrt{\frac{3}{5}} \qquad = \sqrt{\frac{3}{2}}$$

$\therefore$ Radius of circle = AP = $\sqrt{CP^2 - CA^2}$

$$= \sqrt{\frac{5}{2} - \frac{3}{2}} \quad = \sqrt{1} \quad = 1$$

Hence, the plane (2) meets the sphere (1) in a circle of radius unity. Now any sphere through the intersection of (1) and (2) is

$x^2 + y^2 + z^2 - x + z - 2 + k (x + 2y - z - 4) = 0$

$x^2 + y^2 + z^2 + (-1 + k) x + 2ky + (1 - k) z - 2 - 4k = 0$ .... (3)

If the circle of intersection of (1) and (2), is a great circle of sphere (3), then the centre $\left(\frac{1-k}{2}, -k, \frac{k-1}{2}\right)$ of sphere (3) and must lie on the plane (2)

$$\therefore \quad \left(\frac{1-k}{2}\right) + 2\,(-k) - \frac{k-1}{2} - 4 = 0$$

$\Rightarrow$ $- 6k - 6 = 0$ $\Rightarrow$ $-6k = 6$

$\Rightarrow$ $k = -1$

Putting $k = -1$ in eq (3), the required sphere is

$x^2 + y^2 + z^2 - x + z - 2 - (x + 2y - z - 4) = 0$

$\Rightarrow$ $x^2 + y^2 + z^2 - 2x - 2y + 2z + 2 = 0$

Example : Prove that the circles

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0, 5y + 6z + 1 = 0$

and $x^2 + y^2 + z^2 - 3x + 4y + 5z - 6 = 0, x + 2y - 7z = 0,$

lie on the same sphere and find its equation

Solution : Equation of the sphere containing the circle

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0, 5y + 6z + 1 = 0$ is

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + k(5y + 6z + 1) = 0$

or $x^2 + y^2 + z^2 - 2x + (3 + 5k)y + (4 + 6k)z + (k - 5) = 0$ ......(1)

and the equation of the sphere containing the circle

$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0,\ x + 2y - 7z = 0$ is

$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 + k(x + 2y - 7z) = 0$

or $x^2 + y^2 + z^2 + (-3 + k^1)x + (-4 + 2k^1)y + (5 - 7k^1)z - 5 = 0$ .....(2)

If the equation (1) and (2) represent the same sphere them compairing the coefficients of x, y, z we have

(i)	(ii)	(iii)	(iv)
$\frac{-2}{-3+k^1}$	$\frac{3+5k}{-4+2k^1}$	$\frac{4+6k}{-5-7k^1}$	$\frac{k-5}{-6}$

From (i) and (ii) we get

$8 - 4k^1 = -9 + 3k^1 - 15k + 5kk^1$

$7k^1 - 15k + 5kk^1 = 17$ .... (3)

From (i) and (iv), we get

$12 = -3k + 15 + kk^1 - 5k^1$

$3k + 5k^1 - kk^1 = 3$ .... (4)

Multiply (3) by 1 and (4) by 5, we get

$7k^1 - 15k + 5kk^1 = 17$

$25k^1 + 15k - 5kk^1 = 15$

$32k^1 = 32 \Rightarrow k^1 = 1$

Substituting the value of k^1 in equation (3), we get

$7 - 15k + 5k = 17$

$\Rightarrow$ $-10k = 10$

$\Rightarrow \quad k = -1$

Substituting the value of k in eq (1) and k^1 in eq (2), we get the same equation to the sphere namely

$$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0.$$

EXERCISES

1. Find the equation of sphere through the circle

   $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and point (1, 2, 3).

2. Find the equation of the circle lying on the sphere

   $x^2 + y^2 + z^2 - 2x + 4y - 6z + 3 = 0$ and having its centre at (2, 5, 4)

3. Find the equation to the sphere having the circle.

   $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0$, $x + y + z = 3$ as the great circle.

4. Find the centre and radius of the circle

   $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$, $x + 2y + z = 15$.

ANSWERS

1. $3x^2 + 3y^2 + 3z^2 - 2x - 3y - 4z - 22 = 0$
2. $x^2 + y^2 + z^2 - 2x + 4y - 6z + 3 = 0$, $x + 5y - 7z - 45 = 0$
3. $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$
4. (1, 3, 4); $\sqrt{7}$

****

4.3.1 INTERSECTION OF SPHERE AND A STRAIGHT LINE

To find the points where the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ meet the sphere $x^2 + y^2 + z^2 + 2ux + 2vy - 2wz + d = 0$

Solution :

The given line is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \text{ (= r say)} \qquad \text{....... (1)}$$

The sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ .... (2)

Any point on the line (1) is $(lr + x_1, mr + y_1, nr + z_1)$ .... (3)

If it lies on the sphere (2), then

$(lr + x_1)^2 + (mr + y_1)^2 + (nr + z_1)^2 + 2u (lr + x_1) + 2v (mr + y_1)$
$+ 2w (nr + z_1) + d = 0$

or $l^2r^2 + x_1^2 + 2lx_1r + m^2r^2 + y_1^2 + 2mry_1 + n^2r^2 + z_1^2 + 2nrz_1 + 2ulr + 2ux_1$

$+ 2vmr + 2wz_1 + d = 0$

or $r^2\left(l^2 + m^2 + n^2\right) + 2r\left[l\left(x_1 + u\right) + m\left(y_1 + u\right) + n\left(z_1 + w\right)\right] + x_1^2 + y_1^2 + z_1^2 +$

$2ux_1 + 2vy_1 + 2wz_1 + d = 0$

which is quadralic in r and hence gives two values of r. Putting these values of r one by one in eg. (3), we get the two points of intersection of a sphere and straight line.

Example : To find the coordinates of the points where the line $\frac{x+3}{4}=\frac{y+4}{3}=\frac{z-8}{-5}$ intersects the sphere $x^2 + y^2 + z^2 + 2x - 10y = 23$

Solution : The equation of line and sphere are given by

$$\frac{x+3}{4}=\frac{y+4}{3}=\frac{z-8}{-5} \ (= r) \qquad .... (1)$$

$$x^2 + y^2 + z^2 + 2x - 10y - 23 = 0 \qquad .... (2)$$

Any point on the line (1) is (ur - 3, 3r - 4, -5r + 8) .... (3)

It lies on the sphere (2), then

$(ur - 3)^2 + (3r - 4)^2 + (-5r +8)^2 + 2(4r - 3) - 10(3r- 4) - 23 = 0$

$16r^2 + 9r^2 + 25r^2 - 2ur - 2ur - 80r + 8r - 30r + 9+16+ 64 - 6 + 40-23 =0$

50r2 - 150r + 100 = 0

$\Rightarrow \quad r^2 - 3r + 2 = 0 \qquad \Rightarrow r^2 - 2r - r + 2 = 0$

$\Rightarrow \quad r(r - 2) - 1 (r - 2) = 0 \Rightarrow (r - 1) (r - 2) = 0 \Rightarrow r = 1, 2$

Substituting r = 1 and r = 2 one by one in (3) we get

(1, -1, 3) and (5, 2, -2) respectively are the points of intersection of line and sphere.

EXAMPLE : Find the coordinates of the points where the line $\frac{x+2}{4}=\frac{y+9}{3}=\frac{z-8}{-5}$ meets the sphere $x^2 + y^2 + z^2 = 49$

SOLUTION : The given line and sphere is

$$\frac{x+2}{4}=\frac{y+9}{3}=\frac{z-8}{-5} = r \text{ (say))} \qquad .... (1)$$

and $x^2 + y^2 + z^2 = 49$ .... (2)

Any point on the line (1) is (ur - 2, 3r - 9, -5r + 8) .... (3)

It it lies on the sphere (2), then

$(ur - 2)^2 + (3r - 9)^2 + (-5r + 8)^2 = 49$

$16r^2 + 9r^2 + 25r^2 - 16r - 54r - 80r + 4 + 81 + 64 - 49 = 0$

$50r^2 - 150r + 100 = 0 \qquad \Rightarrow r^2 - 3r + 2 = 0$

$\Rightarrow \qquad r = 1, 2$

Substituting r = 1 and r = 2 one by one is (3) we get (2, -6, 3) and (6, -3, 2) are the points of the intersection of sphere and line.

4.3.2 EQUATION OF THE TANGENT PLANE AT A POINT

To find the equation of the tngent plane at the point (x_1, y_1, z_1) to the sphere

i) $x^2 + y^2 + z^2 = a^2$

ii) $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Proof : i) The given sphere is $x^2 + y^2 + z^2 = a^2$ .... (1)

Any line through the point (x_1, y_1, z_1) is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} = r \qquad .... (2)$$

Coordinate of any point on the line are $(lr + x_1, mr + y_1, nr + z_1)$

If it lies on the sphere (2), then

$(lr + x_1)^2 + (mr + y_1)^2 + (nr + z_1)^2 = a^2$

$$\Rightarrow \qquad r^2\left(l^2+m^2+n^2\right)+2r\left(lx_1+my_1+nz_1\right)+x_1^2+y_1^2+z_1^2-a^2=0 \qquad .... (3)$$

which is quadratic in r.

Since (x_1, y_1, z_1) lies on (1)

$\therefore$ $x_1^2 + y_1^2 + z_1^2 - a^2 = 0$ .... (4)

$\therefore$ equation (3) becomes, $r^2\left(l^2 + m^2 + n^2\right) + 2r\left(lx_1 + my_1 + nz_1\right) = 0$ -- (5)

$\therefore$ One value of r is zero. If the line (2) touches the sphere (1), then the line meets the sphere only in one point (two coincident points) so that the two values of r in eq. (5) must be equal. But one value of r is zero, therefore, other value of r must also zero, i.e. coefficient of r = 0

$\Rightarrow$ $lx_1 + my_1 + nz_1 = 0$ ---- (6)

To find the focus of the line (2), we have to elliminated, m, n from (2) and (6). Substituting the value of l, m, n from eq (2) into eq (6), we get

$(x - x_1)\, x_1 + (y - y_1)\, y_1 + (z - z_1)\, z_1 = 0$

$\Rightarrow$ $xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2$

$\Rightarrow$ $xx_1 + yy_1 + zz_1 = a^2$

which is the required equation of the tangent plane at (x_1, y_1, z_1) to the sphere (1).

ii) The given sphere is

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ .... (1)

Any line through (x_1, y_1, z_1) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \text{ (say)} \qquad \text{......... (2)}$$

Any point on line (2) is $(lr + x_1, mr + y_1, nr + z_1)$

If it lies on (1), then we have

$(lr + x_1)^2 + (mr + y_1)^2 + (nr + z_1)^2 + 2u\,(lr + x_1) + 2v\,(mr + y_1) + 2\,w\,(nr + z_1) + d = 0$

$\Rightarrow \quad r^2\left(l^2 + m^2 + n^2\right) + 2r\left[l(u + x_1) + m(v + y_1) + n(w + z_1)\right] x_1^2 + y_1^2 + z_1^2 + 2ux_1 +$

$2vy_1 + 2wz_1 + d = 0$ .... (3)

which is quadratic in r. Since (x_1, y_1, z_1) lies on sphere (1)

$\Rightarrow \quad x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$ .... (4)

$\therefore$ Equation (3) becomes

$\left(l^2 + m^2 + n^2\right)r^2 + 2r\left[l(u + x_1) + m(v + y_1) + n(w + z_1)\right] = 0$ .... (5)

Its one root is zero. Since the line (2) touches the sphere so both the values of r from (5) must be equal. As one value of r is already zero. This implies 2nd value of r from (5) must alsobe zero

$\Rightarrow$ Coefficient of r = 0

i.e. $l(u + x_1) + m(v + y_1) + n(w + z_1) = 0$ .... (6)

Eliminating l, m, n from (2) and (6), the locus of the tangent lines (2) is

$(x - x_1)(u + x_1) + (y - v_1)(v + y_1) + (z - z_1)(w + z_1) = 0$

$\Rightarrow \quad xu + xx_1 - x_1^2 + vy - vy_1 + yy_1 - y_1^2 + wz - wz_1 + zz_1 - z_1^2 = 0$

$\Rightarrow \quad xx_1 + yy_1 + zz_1 + ux + vx + wz = x_1^2 + y_1^2 + z_1^2 + ux_1 + uy_1 + wz_1$

Adding $ux_1 + vy_1 + wz_1 + d$ to both sides, we set

$xx_1 + yy_1 + zz_1 + ux + uy + wz + ux_1 + vy_1 + wz_1 + d = 0$

which is the required equation of the tangent plane at (x, y, z)

Note : The radius of sphere through the poind of contact of the tangent plane is perpendicular to the tangent plane.

4.3.3 TANGENT PLANE PROPERTY

i) If a plane touches a sphere, then $\perp$ distance of the centre from the plane = radius of the sphere

ii) Similarly if a line touches a sphere, then the $\perp$ distance of the centre from the line = radius of the sphere

Example : Show that the plane 2x - 2y + z + 12 = 0 touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and find the point contact.

Solution : The equation of the plane is

$2x - 2y + z + 12 = 0$

and the sphere is $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$

Here 2u = -2, 2v = -4, 2w = 2, d = -3

i.e. u = -1, v = 2, w = 1, d = -3

$\therefore$ Centre (-u, -v, -w) i.e. (1, 2, -1)

and radius = $\sqrt{u^2 + v^2 + w^2 - d}$ = $\sqrt{1+4+1+3}$

= $\sqrt{9}$ = 3

Now the plane 2x - 2y + z + 12 = 0 touches the givn sphere if the length of the $\perp$ from the centre (1, 2, -1) of the sphere to the plane is equal to the radius of the sphere. So we have

$$\frac{(2)(1)-2(2)-1+12}{\sqrt{4+4+1}} = 3 \text{ which is true}$$

Also the point of contact is the foot of the $\perp$ from the centre (1, 2, -1) of the sphere to the plane 2x - 2y + z + 12 = 0

$\therefore$ Equation of the line through (1, 2, -1) and $\perp$ to the sphere

$2x - 2y + 2z + 12 = 0$

are $\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}$ = r

any point on the line is (2r + 1, -2r + 2, r -1)

If this point lies on the plane then

2 (2r + 1) - 2 (-2r + 2) + (r - 1) + 12 = 0

ur + 2 + ur - 4 + r - 1 + 12 = 0

$\Rightarrow$ 9r = -9 $\Rightarrow$ r = -1

Hence point of contact is (-2 + 1, 2 + 2, -1 - 1)

i.e. (-1, 4, -2), which is the required point contact.

Example : Find the equation of the tangent plane to the sphere

$x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$

which are parallel to the plane 2x + y - y = 0

Solution : Equation of the sphere is $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$

Its centre is (2, -1, 3)

and radius = $\sqrt{4+1+9-5}$ = $\sqrt{9}$ = 3

Any plane parallel to the plane 2x + y - z = 0 is 2x + y - z = k

It it touches the sphere then length $\perp$ from the centre of the sphere must be equal to the radius of the sphere.

i.e. $\frac{2(2)+(-1)-(-3)-k}{\sqrt{4+1+1}}=3$

$\Rightarrow$ $\frac{-k}{\sqrt{16}}=3$ $\Rightarrow$ k = $-3\sqrt{6}$ $\Rightarrow$ $k^2 = 54$ $\Rightarrow$ $k = \pm 3\sqrt{6}$

Hence the required equation of the plane are 2x + y - z = $\pm 3\sqrt{6}$

EXERCISES

1. To find the equation of the tangent plane at P(-3, 5, 4) to the sphere $x^2 + y^2 + z^2 + 5x - 7y + 2z - 8 = 0$.
2. Show that the plane $2x + y - z = 12$ touches the sphere $x^2 + y^2 + z^2 = 24$ and find its point of contact.
3. Find the equation of sphere which touches the sphere $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$ at the point (1, 2, -2) and passes through the angle
4. Obtain the equation of the tangent planes to the sphere $x^2 + y^2 + z^2 = 9$ which can be drawn through the line $\frac{x-5}{2} = \frac{y-1}{-2} = \frac{z-1}{1}$.
5. Find the equation of the tangent planes to the sphere $x^2 + y^2 + z^2 + 4x - 6y + 2z - 5 = 0$ at the point (1, 2, 3).

ANSWERS :

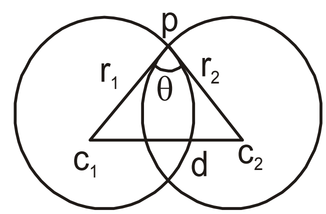
1. $x - 3y - 10z + 58 = 0$
2. (4, 2, -2) and (-4, -2, 2)
3. $4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$
4. $x + 2y + 2z - 9 = 0,\ 2x + y - 2z - 9 = 0$
5. $3x - y + 3z - 7 = 0$

4.3.4 ANGLE OF INTERSECTION OF TWO SPHERES

The angle of intersection of two spheres is the angle between their tangent at a common point of intersection. Also the ratio of the spheres to a common point are $\perp$ to the tangent plane at that point.

4.3.5 FIND THE ANGLE OF INTERSECTION OF TWO SPHERES AT THE COMMON POINT OF INTERSECTION.

Solution : Let C_1, C_2 be the centre of two spheres of radi r_1 and r_2. Let p be the common poinit of intersecton and let d = C_1C_2.



Angle of intersection of two spheres Q

= Angle between the tangent at P

= Angle betwen radii of the two spheres

$$\Rightarrow \quad \text{Cos}\theta = \frac{(c_1p)^2 + (c_2p)^2 - (c_1c_2)^2}{2(c_1p)(c_2p)} = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} \quad \text{.........} \ (1)$$

$$\Rightarrow \quad \theta = \text{Cos}^{-1}\left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$$

4.3.6 CONDITIONS OF ORTHOGONALITY OF TWO SPHERES

If two spheres are orthogonal, then $\theta = 90$

$\Rightarrow$ Cos 90 = 0

Therefore (2) radious to $r_1^2 + r_2^2 - d^2 = 0$

$$\Rightarrow \quad r_1^2 + r_2^2 = d^2$$

i.e., square of distance between the centre of two spheres = sum of square of the radii

To find the condition that the spheres

$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$

and $x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$

may cut orthogonally

Solution : The two spheres are

$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$ .... (1)

and $x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$ .... (2)

If the sphers cut orthogonally, then square of distance between their centres = sum of the spheres of their radii ....(3)

Now the centre of sphere (1) and (2) are $c_1(-u_1, -v_1, -w_1)$ and $c_2(-u_2, -v_2, -w_2)$ and their radii are

$$\sqrt{u_1^2 + v_1^2 + w_1^2 - d_1}\ ,\ \sqrt{u_2^2 + v_2^2 + w_2^2 - d_2}$$

$\therefore$ From eq (3), we have

$$(-u_2 + u_1)^2 + (-v_2 + v_1)^2 + (-w_1 + w_1)^2 + (u_1^2 + v_1^2 + w_1^2 - d_1)$$

$$+ (u_2^2 + v_2^2 + w_2^2 - d_2)$$

$$\Rightarrow \quad u_1^2 + u_2^2 - 2u_1u_2 + v_1^2 + v_2^2 - 2v_1v_2 + w_1^2 + w_2^2 - 2w_1w_2 = u_1^2 + v_1^2 + w_1^2 - d_1$$

$$+ u_2^2 + v_2^2 + w_2^2 - d_2$$

$\Rightarrow$ $-2u_1u_2 - 2v_1v_2 - 2w_1w_2 = -d_1 - d_2$

$\Rightarrow$ $2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

which is the required condition of orthogonality of two spheres.

Example : Find the ange of intersection of the two spheres.

$x^2 + y^2 + z^2 + 2x - 4y - 6z + 10 = 0$ .... (1)

and $x^2 + y^2 + z^2 - 6x - 2y - 2z + 2 = 0$ .... (2)

Solution : Centre of first sphere $c_1(1, 2, 3)$

Centre of second sphere $c_2(3, 1, 1)$

Radius of first sphere $r_1 = \sqrt{1+4+9-10} = \sqrt{4} = 2$

Radius of second sphere $r_2 = \sqrt{9+1+1-10} = \sqrt{9} = 3$

Therefore d = $c_1c_2 = \sqrt{4+1+4} = \sqrt{9} = 3$

Let θ be the angle between the two spheres, then

$$\text{Cos}\,\theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2} = \frac{4+9-9}{2x2x3} = \frac{4}{12} = \frac{1}{3}$$

$\Rightarrow$ $\theta = \cos^{-1}\left(\frac{1}{3}\right)$ is the required angle of intersection of (1) and (2)

Example : Show that the spheres $x^2 + y^2 + z^2 + 6y + 2z + 8 = 0$

and $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$

are orthogonal. Find their plane of intersection.

Solution : The given spheres are

$x^2 + y^2 + z^2 + 2x + 6y + 2z + 8 = 0$ .... (1)

and $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$ .... (2)

Here $u_1 = 0$, $v_1 = 3$, $w_1 = 1$, $d_1 = 8$, $u_2 = 3$, $v_2 = 4$, $w_2 = 2$, $d_2 = 20$

Thus the sphere will be orthogonal if

$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

or if $2(0)(3) + 2(3)(4) + 2(1)(2) = 8 + 20$

or if $0 + 24 + 4 = 28$

or if 28 = 28 which is true

Hence the two spheres are orthogonal.

Also the plane of intersection of these two given sphere is

$x^2 + y^2 + z^2 + 6y + 2z + 8 - (x^2 + y^2 + z^2 + 6x + 8y + 4z + 20) = 0$

$\Rightarrow$ $-6x - 2y - 2z - 12 = 0$

$\Rightarrow$ $3x + y + z + 6 = 0$ is the required equation of plane.

EXERCISES

1. Find the equaton of the sphere through the crcle

$x^2 + y^2 + z^2 -2x + 3y - 4z + 6 = 0$

$3x - 4y + 5z - 15 = 0$

Cutting the spheres $x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ orthogonally.

2. Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the value of the common circle is $\frac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$.

3. Find the radius of the sphere which touches the plane $3x+ 2y- z + 2 = 0$ at the point (1, -2, 1) and also cut orthogonally the shere

$x^2 + y^2 + z^2 -4x + 6y + 4 = 0$.

ANSWERS

1. $5(x^2 + y^2 + z^2) - 13x + 19y - 25z + 45 = 0$

2. Radius = $\sqrt{\frac{63}{2}}$

****

5.1 CONE AND CYLINDER

5.1.1 CONE DEFINITION

The cone is a three dimensional geometric shape and a surface generated by

i) a straight line which passes through a fixed point

ii) intersect a given curve or touches a given surface.

The fixed point is called the vertex and the given curve (or surface) is called the guiding curve of the cone. A straight line passing through the vertex and intersecting the curve or touching the surface is caed a generator.

5.1.2 EQUATION OF CONE WITH VERTEX AT ORIGIN

Find the equation of the cone whose vertex at the origin.

Solution : The equation of any line passing through O(o, o, o)

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n}= r \text{ (say)} \qquad .... (1)$$

Since the line intersect the curve

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vyw + 23wz + d = 0$ -(2)

At any point on the (1) is (rl, rm, rn)

This point will like on the curve (2) if

$a(rl)^2 + b(rm)^2 + c(rn)^2 + 2f(rm)(rn) + 2g(rn)(rl) + 2h(rl)(rm) + 2u(rl)$
$+ 2v(rm) + 2w(rn) + d = 0$

$r^2(al^2 + bm^2 + cn^2 + 2fnm + gnl + 2hlm) + 2r(ul + vm + wn) + d = 0$

for each value of r,

Therefore,

$al^2 + bm^2 + cn^2 + 2fnm + gnl + 2hlm = 0$ --- (3)

$ul + vm + wn = 0$ --- (4)

$d = 0$ --- (5)

From eq (4), it is clear that u = v = w = 0, otherwise the point (l, m, n) lies on the plane ux + vy + wz = 0 which is contradiction

Hence the regd equation of the cone is

$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0$

5.1.3 EQUATION WITH VERTEX (α, β, γ) AND BASIC CONIC

$f(xy) = ax^2 + hxy + by^2 + 2gz + 2fy + c = 0, z = 0$

Solution : Let the equation of any line pasing through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{....... (1)}$$

and the given guiding curve is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c, z = 0$ .... (2)

Since (1) meets the plane z = 0 at the point $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

Since the line (1) intersect the curve (2), we get

$$a\left(\alpha - \frac{l\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{l\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + b\left(\beta - \frac{m\gamma}{n}\right)^2 + 2g\left(\alpha - \frac{l\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0$$

.... (3)

Eliminating l, m, n from the above equation (3), we get the required equaton of the cone as

$$a\left(\alpha - \frac{x-a}{z-\gamma}\gamma\right)^2 + 2h\left(\alpha - \frac{x-a}{z-\gamma}\gamma\right)\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right) + b\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right)^2 + 2g\left(\alpha - \frac{x-a}{z-\gamma}\gamma\right)$$

$$+ 2f\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right) + c = 0$$

$$\Rightarrow \quad a(\alpha z - \gamma x)^2 + 2h(\alpha z - \gamma x)(\beta z - \gamma y) + b(\beta z - \gamma y)^2 + 2g(\alpha z - \gamma x)(z - 8)$$

$$+ 2f(\beta z - \gamma x)(z - \gamma) + c(z - \gamma)^2 = 0$$

Example : Find the equation of the cone whose vertex is the point (-1, 1, 2) and whose guiding curve is

$3x^2 - y^2 = 1, z = 0$

Solution : The guiding curve is $3x^2 - y^2 = 1, z = 0$

Any line through the point (-1, 1, 2) is

$$\frac{x-(-1)}{l} = \frac{y-1}{m} = \frac{z-2}{n} \qquad \text{....... (2)}$$

i.e., $x = -1 - \frac{2l}{n}, y = -1 - \frac{2m}{n}$

This point lies on the conic (1)

$\therefore \qquad 3\left(-1-2\frac{l}{n}\right)^2-\left(1-2\frac{m}{n}\right)^2=1$ .... (3)

Eliminating l, m, n from (2) and (3) we get

$$3\left[1+2\left(\frac{x+1}{z-2}\right)\right]^2-\left[1-2\left(\frac{y-1}{z-2}\right)\right]^2=1$$

Example : Find the equation to the cone with vertex at origin at origin which passes through the curve

$x^2 + y^2 + z^2 - x - 1 = 0,\ x^2 + y^2 + z^2 + y - 2 = 0$

Solution : Since the equation of the guiding curve are

$x^2 + y^2 + z^2 - x - 1 = 0 \Rightarrow x^2 + y^2 + z^2 - tx - t^2$ where $x = 1$ ...... (1)

and $x^2 + y^2 + z^2 + y - 2 = 0 \Rightarrow x^2 + y^2 + z^2 - ty - 2t^2 = 0$ ...... (2)

Eliminating t from (1) and (2) we get

$- tx - t2 - ty + 2t^2 = 0$

$\Rightarrow \quad t^2 - tx - ty = 0 \quad \Rightarrow \quad t(t - x - y) = 0$

But $t \neq 0$,

$\therefore \quad t - x - y = 0 \quad \Rightarrow \quad t = x + y$

Putting the value of t in eq (1), we get

$\Rightarrow \quad x^2 + y^2 + z^2 - x^2 - xy - x^2 - y^2 - 2xy = 0$

$\Rightarrow \quad z^2 - x^2 - 3xy = 0 \quad \Rightarrow \quad x^2 - z^2 + 3xy = 0$

This is the required equation of the cone

Example : Find the equation of cone whose vertex is (1, 2, 3) and the guiding curve is the circle

$x^2 + y^2 + z^2 = 4,\ x + y + z = 1$

Solution : The given curve is

$x^2 + y^2 + z^2 = 4,\ x + y + z = 1$ .... (1)

Any line through the point (1, 2, 3) is

$\Rightarrow$ $x^2 + y^2 + z^2 + ty - 2t^2 = 0$ --- (2)

Eliminating t from (1) and (2), we get

$- tx - t^2 - ty + 2t^2 = 0$

$\Rightarrow$ $t^2 - tx - ty = 0$ $\Rightarrow$ $t (t - x - y) = 0$

But $t \neq 0$,

$\therefore$ $t - x - y = 0$ $\Rightarrow$ $t = x + y$

Putting the value of t in eq (1), we get

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = r \qquad \text{....... (2)}$$

Any point on the line is (lr + 1, mr + 2, nr + 3) .... (3)

Since the point (lr + 1, mr + 2, nr + 3) lies on (1), therefore

$(lr + 1)^2 + (mr + 2)^2 + (nr + 3)^2 = 4$ .... (4)

and $(lr + 1) + (mr + 2) + (nr + 3) = 1$ .... (5)

From (5), we have

$$r(l + m + n) = -5 \Rightarrow r = \frac{-5}{l+m+n} \qquad \text{....... (6)}$$

we have from eq (4), we have

$(l^2 + m^2 + n^2)\, r^2 + (l + 2m + 3n)\, r = -10$

$$\Rightarrow \quad (l^2 + m^2 + n^2)\left(\frac{-5}{l+m+n}\right)^2 + 2(l+2m+3n)\left(\frac{-5}{l+m+n}\right) = -10$$

c of (6)

$$\frac{25\left(l^2+m^2+n^2\right)}{(l+m+n)^2}-10\frac{(l+2m+3n)}{(l+m+n)}=-10$$

$\Rightarrow$ $25(l^2 + m^2 + n^2) -10 (l + 2m + 3n) (l + m + n) = -10(l + m + n)^2$

$\Rightarrow$ $25(l^2 + m^2 + n^2) -10l^2 - 10lm - 10ln - 20ml - 20m^2 - 20mn - 30nl - 30nm - 30n^2 = -10 (l^2 + m^2 + n^2 + 2lm + 2mn + 2nl)$

$\Rightarrow$ $5l^2 + 3m^2 + n^2 - 2lm - un - 6mn = 0$ .... (7)

Eliminating l, m, n from (2) and (7), we get

$5(x-1)^2 + 3(y-2)^2 + (z-3)^2 -(2x-1)(y-2) - 4(z-3)(x-1)- 6(y-2)(z-3) = 0$

or $5(x^2 - 2x + 1) + 3(y^2 - 4y + 4) + (z^2 - 6z + 9) -2 (xy - 2x -y + x)$
$-4 (xz - z - 3z + 3) - 6(yz - 3y - 2z + 6) = 0$

$\Rightarrow$ $5x^2 + 3y^2 + z^2 - 10x + 5 + 12 - 12y - 6z + 9 - 2xy + 4x + 2y - 4 - 4xz + 4z + 12x - 12 - 6yz + 18y + 12z - 36 = 0$

$5x^2 + 3y^2 + z^2 - 2xy - 4xz - 6yz + 6x + 8y + 10z = 26$

which is th required equation of the cone.

Example : The section of a cone whose guiding curve is the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ by the plane x = 0 is a rectangular hyperboa. Prove

that the locus of the vertex is the surface $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$

Solution : Let (α,β,γ) be the vertex of the cone. Then the equations of a generator of the cone are

$$\frac{x-\alpha}{l}=\frac{y-m}{m}=\frac{z-\gamma}{n} \qquad .... (1)$$

This line will meet to the plane z = 0 in the point given by $\left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n},0\right)$. The point $\left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n},0\right)$ will lie on the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ if

$$\frac{1}{a^2}\left(\alpha-\frac{x-\gamma}{z-\gamma}\gamma\right)^2+\frac{1}{b^2}\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)^2=1$$

$$\Rightarrow \quad \frac{1}{a^2}(\alpha z-\gamma x)^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-8)^2$$

$$\Rightarrow \quad \frac{\gamma^2 y^2}{b^2}+\left(\frac{\alpha^2}{a^2}+\frac{p^2}{b^2}-1\right)z^2-\frac{2\beta\beta\gamma y}{b^2}+2\gamma\gamma-\gamma^2=0$$

The section will represent a rectangular hyperbola in yz plane if

Coefficient of y^2 + coefficient of z^2 = 0

$$\Rightarrow \quad \frac{\gamma^2}{b^2}+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right)=0$$

Hence the locus of (α,β,γ) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}=1$$

$$\Rightarrow \quad \frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$$

EXAMPLE : Find the equation to the quadratic cone which passes through the three coordinates axes and the three mutually perpendicular lines

$$\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}, \frac{x}{1}=\frac{y}{-1}=\frac{z}{-1}, \frac{x}{5}=\frac{y}{4}=\frac{z}{1}$$

Solution : Any cone through the coordinates axes is

fyx + gzx + hxy = 0

Since it pass through the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$

Therefore, the direction ratios of the generator satisfy the equation of the cone,

f(-2)(3) + g(3)(1) + h(1)(-2) = 0

$\Rightarrow$ -6f + 3g - 2h = 0 --- (2)

Similarly as (1) passes through $\frac{x}{1}=\frac{y}{-1}=\frac{z}{-1}$, we have

f(-1(-1) = g((-1)(-1) + h(1)(-1) = 0

$\Rightarrow$ f - g - h = 0 --- (3)

Solving (2) and (3) we have

$$\frac{f}{(-3)(-1)-(-1)(-2)}=\frac{g}{(1)(2)-(6)(-1)}=\frac{h}{6(-1)-(1)(-3)}$$

$\Rightarrow$ $\frac{f}{5}=\frac{g}{8}=\frac{h}{-3}$ --- (4)

Now ellimінating f, g, h from (1) and (4) we get

5yz + 8zx - 3xy = 0

This is the required ratios of the third generator $\frac{x}{5}=\frac{y}{4}=\frac{z}{1}$ satisfy the equation of cone.

f(-2)(3) + g(3)(1) + h(1)(-2) = 0

$\Rightarrow$ -6f + 3g - 2h = 0 --- (2)

Similarly as (1) passes through $\frac{x}{1}=\frac{y}{-1}=\frac{z}{-1}$, we have

f(-1)(-1) + g(-1)(1) + h(1)(-1) = 0

$\Rightarrow$ f - g- h = 0 ---- (3)

Solving (2) and (3), we have

$$\frac{f}{(-3)(-1)-(-1)(2)}=\frac{g}{(1)(2)-(6)(-1)}=\frac{h}{6(-1)-(1)(-3)}$$

$\Rightarrow$ $\frac{f}{5}=\frac{g}{8}=\frac{h}{-3}$ --- (4)

Now elluminating f, g, h, from (1) and (4) we get

5yz + 8zx - 3xy = 0

This is the required ratios of the third generator $\frac{x}{5}=\frac{y}{4}=\frac{z}{1}$ satisfy the equation of cone

EXERCISES

1. Find the equation of the cone whose vertex is (α,β,γ) and whose base is

   i) y^2 = 4ax, z = 0 ii) z^2 = 4ax, y = 0

iii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ iv) $ax^2 + by^2 = 1$, $z = 0$

2. Find the equation to the cone with vertex at origin which passes through the curve

   $ax^2 + bx^2 = 2z$, $lx + my + nz = p$

3. Find the equation of the cone whose vertex is the point (1, 1, 3) and which pass through the ellipse

   $4x^2 + z^2 = 1$, $y = 4$

4. Find the equation of the cone whose vertex is the point (3, 1, 2) and whose guiding lines pass through the ellipse $zx^2 + 3y^2 = 1$, $z = 0$

ANSWERS

1. i) $(\beta z-\gamma y)^2 = 4a(\alpha z-\gamma x)(z-y)$

   ii) $(\beta z-\gamma y)^2 = 4a(\beta x-\gamma y)(\beta-y)$

   iii) $\left(\frac{\alpha z-\gamma x}{a^2}\right)+\left(\frac{\beta z-\gamma y}{b^2}\right)=(z-\gamma)^2$

   iv) $a(\alpha(-\gamma y)^2+b(\beta(-\gamma y)^2=(y-\gamma)^2$

2. $p(ax^2 + by^2) = 2z(lx + my + nz)$

3. $2x^2 + 4y^2 + 3y^2 + 6yz + 8xy - 32x - 34y - 24z + 69 = 0$

4. $2x^2 + 3y^2 + 5z^2 - 3yz - 6xz + z - 1 = 0$

   General Second degree equation of second degree to represent a cone

5.1.4 TO FIND THE CONDITION FOR THE GENERAL EQUATION OF SECOND DEGREE TO REPRESENT A CONE

Let the general equation of second degree

$ax^2 + by^2 + cz^2 + 2fyz + gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ .....(1)

represents a cone whose vertex is the poing (α, β, γ)

Then transforming the origin to (α, β, γ)

Than equation (1) reduces to

$$a(x+\alpha)^2 + b(y+\beta)^2 + c(z+\gamma)^2 + 2f(y+\beta)(z+\gamma) + 2h(x+\alpha)(y+\beta)$$

$$+2u(x+\alpha) + 2v(y+\beta) + 2w(z+\gamma) + d = 0$$

or $ax^2 + by^2 + cz^2 + 2fyz + gzx + 2hxy + \; + 2(a\alpha + h\beta + g\gamma + u)$

$$+2(h\gamma + b\beta + f\gamma + v) + 2(g\gamma + f\beta + c\gamma + w)z + 2u\gamma + 2v\beta + 2w\gamma + d$$

$$+ a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta f + 2g\gamma g + 2h\alpha h = 0$$

Since this represents a cone whose vertex is the orign, it must be a homogenous equation. Therefore, we have

$a\alpha + h\beta + g\gamma + u = 0$ .... (2)

$h\alpha + b\beta + f\gamma + v = 0$ .... (3)

$g\gamma + f\beta + c\gamma + w = 0$ .... (4)

$$+2u\alpha + 2v\beta + 2w\gamma + d + a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta f + 2g\gamma g + 2h\alpha h = 0$$

or $+2(a\alpha + h\beta + g\gamma + u) + 2(h\gamma + b\beta + f\gamma + v) + 2(a\alpha + f\beta + c\gamma + w)$

$$+u\alpha + v\beta + w\gamma + d = 0$$

Using (2) (3) & (4) in equation (x), we have

$u\alpha + v\beta + w\gamma + d = 0$ .... (5)

Eliminating α, β, γ from (2), (3), (4) and (5) we have

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = u$$

which is the required condition

Remark : As working rule, a convenient way of getting the equations that determine the vertex if the cone.

$$ax^2 + b\beta^2 + c\gamma^2 + 2f\beta f + 2g\gamma g + 2h\alpha h + 2u\alpha + 2v\beta + 2w\gamma + d = 0$$

is to first make it homogeneous involving a new variable t expression with respect to x, y and z respectively, followed by the substitution of 1 for t.

Thus if

$F(x, y, z) = ax^2 + by^2 + cz^2 + dt^2 + 2fyz + 2gzx + 2hxy + 2uxt + 2vyt + 2wzt$, then

$$Fx = \frac{\partial F}{\partial x} = 2(ax + hy + gz + ut)$$

$$Fy = \frac{\partial F}{\partial y} = 2(hx + by + fz + ut)$$

$$Fz = \frac{\partial F}{\partial z} = 2(gx + fy + cz + wt)$$

So $Fx = Fy = Fz$ with $t = 1$, gives

$ax + hy + gz + u = 0$

$hx + by + fz + v = 0$

$gx + fy + cz + w = 0$

These equations on solving gives the vertex.

Example : Prove that the equation

$7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26x - 2y + 2z - 17 = 0$

represents a cone whose vertex in (1, -2, 2)

Solution : The given equation is

$7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26x - 2y + 2z - 17 = 0$ ...... (1)

We introduce proper powers of t for making eq. (1) homogeneous

$F(x, y, z) = 7x^2 + 2y^2 + 2z^2 - 10zx + 10xy + 26x - 2y + 2z - 17 = 0$

Therefore,

$Fx = 14x - 10z + 10y + 26t = = 14x - 10z + 10y + 26$ [$\because$ t = 1]

$Fy = 4y + 10x - 2t = 4y + 10x - 2$ [$\because$ t = 1]

$Fz = 4z - 10x + 2t = 4z - 10x + 2$ [$\because$ t = 1]

and $Ft = 26x - 2y + 2x - 34t = 26x - 2y + 2z - 34$ [$\because$ t = 1]

Putting Fx = 0, Fy = 0 and Ft = 0, we have

$14x - 10z + 10y + 26 = 0 \Rightarrow 7x + 5y - 5z + 13 = 0$ .... (2)

$4y + 10x - 2 = 0 \Rightarrow 5x + 2y - 1 = 0$ .... (3)

$4z - 10x + 2 = 0 \Rightarrow -5x + 2z + 1 = 0$ .... (4)

and $26x - 2y - 2z - 34 = 0 \Rightarrow 13x - y + z - 17 = 0$ .... (5)

Adding (3) and (4) we get

$y + z = 0$ .... (6)

Multiplying eq (2) by 13 and (5) by 7, we get

$$\begin{array}{l} 91x + 65y - 65z + 169 = 0 \\ 91x - 7y \quad + 7z \quad - 119 = 0 \\ - \quad + \quad - \quad + \\ \hline \qquad 72y - 72z + 288 = 0 \\ \hline \end{array}$$

$\Rightarrow$ $y - z = -4$ .... (7)

Adding (6) and (7), we get

$2y = -4 \Rightarrow y = -2$

Using the value of 'y' in eq (6) we get $z = 2$

From (4), we have

$-5x + 2x \times 2 + 1 = 0$

$\Rightarrow$ $-5x + 4 + 1 = 0$

$\Rightarrow$ $-5x + 5 = 0$

$\Rightarrow$ $x = 1$

Substituting these values, we have

$13(1) - (-2) + 2 - 17 = 0 \Rightarrow 17 - 17 = 0$ which is true

Hence the given equation represents a cone and its vertex is (1, -2, 2)

EXERCISES

1. Show that the equation

   $x^2 - 2y^2 + 3z^2 - 4xy + 5yz + 6zx + 8x - 19y - 2z - 20 = 0$

   represents a cone with vertex (1, -2, 3).

2. Show that the equation

   $2x^2 + 2y^2 + 7z^2 - 10yz - 10zx + 2x + 2y + 26z - 17 = 0$

   represents a cone whose vertex is (2, 2, 1).

3. Prove that the equation

   $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$

   represents a cone

   if $\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d$

****

5`.2.1 INTERSECTION OF A CONE WITH A PLANE

To find the angle between the lines in which the plane

$ux + vy + wz = 0$ .... (1)

cuts the cone

$f(x, y, z) + ax^2 + by^2 + cz^2 + 2fz + 2gzx + 2hxy = 0$ .... (2)

Let the plane (1) cuts the cone (2) in the line whose equations are given by

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad \text{........ (3)}$$

Since the line (3) lies on (2) and (1) both, we have

$f(l, m, n) = al^2 + bm^2 + 2fmn + 2gnl + 2hlm = 0$ .... (4)

and $ul + vm + wn = 0$ .... (5)

Eliminating n from (4) and (5) we get

$$al^2 + bm^2 + c\left(\frac{ul + vm}{-w}\right)^2 + 2fm\left(\frac{ul + vm}{-w}\right) + 2gl\left(\frac{ul + vm}{-w}\right) + 2hml = 0$$

$$\Rightarrow \quad \frac{l^2}{m^2}\left(cu^2 + aw^2 - 2gwu\right) + \frac{2l}{m}\left(hw^2 + cuv - fwu - gvw\right) + (bw^2 + cw^2 - 2fvw) = 0$$

.... (6)

Equation (6) is quadratic in $\frac{l}{m}$, which gives two values of $\frac{l}{m}$ (real and distinct, concident or imaginary) corresponding to which there are two lines of intersection of the plane and the cone.

Note : If θ is the angle between the generating lines in which a plane cuts a cone and l_1, m_1, n_1 and l_2, m_2, n_2 by the direction ratios of these lines then

$$\cos\theta = \frac{l_1 l_2 + m_1 n_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

These lines will be right angle if $l_1 l_2 + m_1 n_2 + n_1 n_2 = 0$

Example: Find the equation of the lines in which the plane

$2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$

Also find the angle between them.

Solution : The given plane is $2x + y - z = 0$ .... (1)

and the cone is $4x^2 - y^2 + 3z^2 = 0$ .... (2)

Suppose line of section be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ .... (3)

Since it lies on the plane (1), then

$2l + m - n = 0$ .... (4)

Also the line (3) lies on the cone (2), therefore the direction cosine's satisfy the equation of the cone. Thus

From (4), we have $n = 2l + m$

Substituting this value of 'n' in eq (5), we get

$4l^2 - m^2 + 3(2l + m)^2 = 0$

$4l^2 - m^2 + 3(4l^2 + 4lm + m^2) = 0$

$\Rightarrow \quad 16l^2 + 2m^2 + 12lm = 0$

$\Rightarrow \quad 8l^2 + m^2 + 6lm = 0$

$\Rightarrow \quad (2l + m)(4l + m) = 0$

Therefore, either $2l + m = 0$ or $4l + m = 0$

i.e., $2l + m + o.n = 0$ or $4l + m + o.n = 0$

Also from (4), $2l + m - n = 0$ and $2l + m - n = 0$

Solving by cross multiplication, we set

$$\frac{l}{-1-0} = \frac{m}{0+2} = \frac{n}{2-2} \text{ and } \frac{l}{-1} = \frac{m}{4} = \frac{n}{2}$$

Thus the dc's of the two lines of intersection are proportional to 1, -2, 0 and -1, 4, 2

Substituting these values in eq (3), we get the required lines of intersection then

$$\cos\theta = \frac{(1)(-1)+(-2)(4)+(0)(2)}{\sqrt{1+4+0}\sqrt{1+16+4}}$$

$$= \frac{9}{\sqrt{5}\sqrt{21}}$$

$$\Rightarrow \quad \cos\theta = \frac{9}{\sqrt{105}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{9}{\sqrt{105}}\right)$$

Example : Prove that the plane $ax + by + cz = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

Solution : The given equation of the plane is

$ax + by + cz = 0$ .... (1)

and the cone is $xy + yz + zx = 0$ .... (2)

compare (2) with $ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0$, we have

$a = 0, b = 0, c = 0 \Rightarrow a + b + c = 0$

Therefore the cone (2) has three mutually perpendicular generators. The plane (1) through the vector (0, 0, 0) lies on the cone (2), if

$bc + ca + ab = 0$

$\Rightarrow \quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ which is the required equation

EXERCISES

1. Find the angle between the lines of section of the plane

   $6x - 10y - 7z = 0$ and cone $108x^2 - 20y^2 - 7z^2 = 0$.

2. If the plane $2x - y + z = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines, then find the value 'c'.

3. Show that the angle between the lines given by

   $x + y + z = 0$, $ayz + bzx + cxy = 0$ is $\frac{\pi}{2}$

   if $a + b + c = 0$, but $\frac{\pi}{3}$ if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

4. Find the angle between the lines of section of the following cones.

   i) $3x + y + 5z = 0$, $6yz - 2zx + 5xy = 0$

   ii) $2x - 3y + z = 0$, $3x^2 - 5y^2 + 36zy - 20zx - 2xy = 0$

5. Prove that the plane $lx + my + ny = 0$ cuts the cone

$(b - c)x^2 + (c - a)y^2 + (a - b)z^2 + 2fy + 2gzx + 2hxy = 0$ in perpendicular lines if $(b - c)l^2 + (c - a)m^2 + (a - b)n^2 + 2fmn + 2gnl + 2hlm = 0$.

ANSWERS :

1. $\cos^{-1}\left(\frac{16}{21}\right)$

2. $c = 2$

4. i) $\cos^{-1}\left(\frac{1}{6}\right)$ ii) $\cos^{-1}\left(\frac{5}{\sqrt{39}}\right)$

******

5.3.1 INTERSECTION OF CONE AND LINE, A TANGENT LINE AND TANGENT PLANE, CONDITION OF TANGENCY, RECIPROCAL CONE

To find the equation of the tangent plane at the point (x_1, y_1, z_1) to the cone

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$

Solution : Any line through (x_1, y_1, z_1) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \text{ (say)} \qquad \text{....... (1)}$$

Any point on the line is $(lr + x_1, mr + y_1, nr + z_1)$

Now if it lies on the cone

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ .... (2) then

$a(lr + x_1)^2 + b(mr + y_1)^2 + c(nr + z_1)^2 + 2f(mr + y_1)(nr + z_1)$
$+ 2g(nr + z_1)(lr + x_1) + 2h(lr + x_1)(mr + y_1) = 0$

or $(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm)r^2 + 2 \ [l(ax_1+hy_1+gz_1)]$
$+ m (hx_1 + by_1 + fz_1) + n [gx_1 + fy_1 +(z_1)] r + ax_1^2 + by_1^2 + cz_1^2$
$+ 2fy_1z_1 + 2gz_1x_1, 2hx_1y_1) = 0$ .... (3)

Since the point (x_1, y_1, z_1) lies on the cone

$\therefore \quad ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0$ .... (4)

Thus eq (3) becomes

$(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm)r^2 + 2[l(ax_1+hy_1+gz_1)$
$+ m (hx_1 + by_1 + fz_1) + n(gx_1 + fy_1 + cz_1)] r = 0$ --- (5)

This is quadratic in r, so the straight line (1) intersects the given cone at the two points, one of them is the point (x_1, y_1, z_1). The straight line (1) will be the tangent to the cone, if another point of intersection coincides with (x_1, y_1, z_1), for which both values of r must be equal to zero. This will require

$$l(ax_1 + hy_1 + gz_1) + m(hx_1 + by_1 + fz_1) + n(gx_1 + fy_1 + cz_1) = 0$$

Locus of the tangent line (1) is obtained by elliminating l, m, n between (1) and (6) which is the requird tangent plane.

Therefore, the equation to the tangent plane at the point (x_1, y_1, z_1) is

$$(x-x_1)(ax_1+hy_1+gz_1) + (y-y_1)(hx_1+by_1+ fz_1) + (z-z_1)(gx_1+fy_1+cz_1) = 0$$

$$(ax_1+hy_1+gz_1)x + (hx_1+by_1+ fz_1)y + (gx_1+fy_1+cz_1) z = 0 \quad \text{---- (7)}$$

$$\left[\because ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0\right]$$

5.3.2 CONDITION OF TANGENCY OF A PLANE AND A CONE :

To find the condition that the plane $lx + my + nz = 0$ may touch the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Notations : In the equation of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

We will use the following notations

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & g & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

2) A, B, C, F, G, H are the cofactors of a, b, c, f, g, h respectively in D so that

$A = bc - f^2$, $B = ca - g^2$, $C = ab - h^2$

$B = gh - af$, $G = hf - bg$, $H = fg - ch$

3) $BC - F^2 = aD$, $CA - g^2 = bD$, $ab - h^2 = cD$ etc

4) $$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & o \end{vmatrix} = [Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv$$

Proof : The given plane is $lx + my + nz = 0$ .... (1)

Let this plane touches the cone at (x_1, y_1, z_1). Then the equation of the tangent plane at (x_1, y_1, z_1) is

$$x(ax_1 + hy_1 + gz_1) + y(hx_1 + by_1 + fz_1) + z(gx_1 + fy_1 + cz_1) = 0 \quad ...... (2)$$

Comparing the coefficients in (1) and (2), we have

$$\frac{ax_1 + hy_1 + gz_1}{l} = \frac{hx_1 + by_1 + fz_1}{m} = \frac{gx_1 + fy_1 + cz_1}{n} = k \text{ (say)}$$

Therefore,

$ax_1 + hy_1 + gz_1 - lk = 0$ ...... (3)

$hx_1 + by_1 + fz_1 - ml = 0$ ...... (4)

$gx_1 + fy_1 + cz_1 - nk = 0$ ...... (5)

Since the point (x_1, y_1, z_1) lies on the plane (1), therefore

$lx_1 + my_1 + nz_1 - ok = 0$ ...... (6)

Eliminating x_1, y_1, z_1, k from (3), (4), (5), (6) with the help of determinants, we have

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & o \end{vmatrix} = 0$$

$\Rightarrow$ $Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$ ...... (7)

Where a, b, c, f, g, h, are the cofactors of corresponding small letters in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Thus (7) is the required condition of the plane (1) to touch cone

Reciprocal Cone

The locus of the normals to the tangent planes through the vertex of the cone is another cone called the reciprocal cone of the given cone.

5.3.4 EQUATION OF THE RECIPROCAL CONE

To find the equation of the cone reciprocal to the cone

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

Solution : The given cone is

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ ...... (1)

Any tangent plane to (1) is $lx + my + nz = 0$ ...... (2)

Where $Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$ ...... (3)

Now the equation of the line through the vertex O(o, o, o) of (1) and normal to the tangent plane (2) are

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n} \Rightarrow \frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \text{...... (4)}$$

Therefore locus of (4) is

$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$ ...... (5)

Which is the required equation of the reciprocal cone where A, B, C, F, G, H, are the cofactors of corresponding small letters in the determinant

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Method to find the reciprocal cone of a given cone

1) Compare the given equation of a given cone with $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ and find a, b, c, f, g, h
2) Find $A = bc\text{-}f^2$, $B = ca\text{-}g^2$, $C = ab\text{-}h^2$, $F = gh - af$, $G = hf\text{-}bg$, $H = fg\text{-}ch$
3) Then reciprocal cone is $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$.

Example : Prove that the cone $ax^2 + by^2 + cz^2 = 0$ and

$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} = 0$ are reciprocal.

Solution : The equation of first cone is $ax^2 + by^2 + cz^2 = 0$

Compare it with $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

We get a = a, b = b, c= c, f = o, g = o, h = o

Therefore $A = bc - f^2 = bc$, $B = ca - g^2 = ca$, $C = ab - h^2 = ab$ and
$F = gh - af = 0$, $G = hf - bg = 0$, $H = fg - ch = 0$

Therefore, the reciprocal cone of (1) is

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

i.e., $bcx^2 + cay^2 + abz^2 = 0$

Dividing through by abc, we get

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} = 0 \text{ which is the second cone}$$

Further, the reciprocal cone $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} = 0$ is

$ax^2 + by^2 + cz^2 = 0$

Hence the given cones are reciprocal.

Example : Prove that general equation of the cone which touches three coordinates plane is

$$\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0 .$$

Solution : Since we know that the general equation of the cone having the coordinates axes as its generators is of the form

$fyz + gzx + hxy = 0$

$\Rightarrow \quad 2fyz + 2gzx + 2hxy = 0 \qquad \text{...... (1)}$

Since the reciprocal cone will touch the planes perpendicular to coordinates axes so the cone touching the coordinate planes is the reciprocal cone of (1)

Since the reciprocal cone of (1) is

$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

Where A, B, C, F, G, H are the cofactors of corresponding small letters in the determinant

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Here a = O, b = 0, c = 0

Therefore A = o - f^2 = - f^2, B = o - g^2 = $-g^2$, C = o - h^2 = $-h^2$

and F = gh - o = gh, G = hf - o = hf, H = fg - o = fg

Therefore the equation of the reciprocal cone of (1) is

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

$-f^2x^2 - g^2y^2 - h^2z^2 + 2ghyz + 2hfzx + 2fgxy = 0$

$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx + 2fgxy = 4fgxy$

$$\Rightarrow \quad (fx + gy - hz)^2 = 4fgxy \Rightarrow fx + gy - hz = \pm 2\sqrt{fgxy}$$

$$\Rightarrow \quad fx + gy \pm 2\sqrt{fgxy} = hz$$

$$\Rightarrow \quad \left(\sqrt{fx} \pm \sqrt{gy}\right)^2 = \pm\left(\sqrt{hz}\right)^2$$

$$\Rightarrow \quad \sqrt{fx} \pm \sqrt{gy} = \pm\sqrt{hz}$$

Hence $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$.

Example : Prove that the perpendicular drawn from the origin to the tangent planes to the cone

$3x^2 + 4y^2 + 5z^2 + 2yz + 4zx + 6xy = 0$ lies on the cone

$19x^2 + 11y^2 + 3z^2 + 6yz - 10zx - 26xy = 0$.

Solution : The given equation of the cone is

$3x^2 + 4y^2 + 5z^2 + 2yz + 4zx + 6xy = 0$ ....(1)

Compairing (1) with the general equation of the cone

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, we get

a = 3, b = 4, c = 5, f = 1, g = 2, h = 3

Therefore, A = bc - f^2 = 19, B = ca - g^2 = 11, C = ab - h^2 = 3 and F = gh - af = 3, G = hf - bg = -5, H = fg - ch = -13

Therefore, the required reciprocal cone is

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

i.e., $19x^2 + 11y^2 + 3z^2 + 6fyz - 10zx - 26xy = 0$

EXERCISES

1. Find the equation of the plane which touches the cone $x^2 + 2y^2 - 3z^2 - 2yz + 5zx + 3xy = 0$ along the generator whose direction cosmies are proportional to 1, 1, -1.
2. Find the reciprocal cone of the cone $ax^2 + by^2 + cz^2 = 0$
3. Show that the equation $a^2x^2 + b^2y^2 + c^2z^2 + 2bcyz + 2cazx + 2abxy = 0$ represent a cone which touches the coordinate planes.

ANSWERS

1. $\frac{9}{2}(y+z)=0$.

2. $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$.

5.3.5 ENVELOPING CONE

The locus of the tangent lines drawn from a given point to a given surface (sphere) is called the enveloping cone or the tangent cone of the surface (sphere). The given point called the vertex of the enveloping cone.

Equation of the enveloping cone

To find the equation of the enveloping cone of the sphere $x^2+y^2+z^2=a^2$ with vertex at the point (α, β, γ).

Solution : Any line through the period (α, β, γ) is given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \text{ (say)} \qquad \text{........... (1)}$$

Any point on (1) is $(\alpha+lr, \beta+mr, \gamma+nr)$

This point lie on the given sphere if

$$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2=a^2$$

$$\Rightarrow \qquad \alpha^2+l^2r^2+2\alpha l+\beta^2+m^2r^2=2\beta mr+\gamma^2+n^2r^2+2\gamma nr=a^2$$

$$(l^2+m^2+r^2)r^2+2(\alpha l+\beta m+\gamma n)r=\alpha^2+\beta^2+\gamma^2-a^2=0 \qquad \text{--- (2)}$$

Equation (2) is quadratic in r, giving two values of r corresponding to which there are two points common to the sphere and the line (1). Now if the line (1) to be the tangent to the sphere, them both the values of r given by the equation (2) must be equal. Therefore,

$$(\alpha l+\beta m+\gamma n)^2=(l^2+m^2+r^2)(\alpha^2+\beta^2+\gamma^2-a^2)$$

The locus of the tangent lines is obtaied by elimiating l, m, n from (1) and (2). Thus the required locus is

$$[\alpha(x-\alpha)+\beta(y-\beta)+\lambda(z-\gamma)]^2=[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]$$

$$(\alpha^2+\beta^2+\gamma^2-a^2) \qquad \text{----- (4)}$$

If we use the notations :

$S = x^2 + y^2 + z^2 - a^2$

$S_1 = \alpha^2+\beta^2+\gamma^2-a^2$

$T = \alpha x+\beta y+\gamma z-a^2$

Then the equation (4) may be written as

$$(T - S_1)^2 = (S - 2T + S_1)S_1 \Rightarrow T^2 = SS_1 \qquad \Rightarrow T^2 = SS_1$$

$\Rightarrow$ $\alpha x + \beta y + \gamma z - a^2 = (x^2 + y^2 + z^2 - a^2)(\alpha^2 + \beta^2 + \gamma^2 - a^2)$

is the required equation of the enveloping cone.

Example : Find the enveloping cone of the sphere $x^2 + y^2 + z^2 + 2x - 2y = 2$ with its vertex at (1, 1, 1).

Solution : The equation of the sphere is $x^2 + y^2 + z^2 + 2x - 2y = 2$ and vertex (1, 1, 1)

Here $S = x^2 + y^2 + z^2 + 2x - 2y - 2 = 0$ and $\alpha = \beta = \gamma = 1$

Therefore

$S_1 = (1)^2 + (1)^2 + (1)^2 + 2(1)^2 - 2(1) - 2 = 1$

and $T = x(1) + y(1) + z(1) + (x - 1) - (y + 1) - 2 = 2x + z - 2$

Therefore equation of enveloping cone is $SS_1 = T^2$

$\Rightarrow$ $(x^2 + y^2 + z^2 + 2x - 2y = 2)(1) = (2x + z - 2)2$

$\Rightarrow$ $x^2 + y^2 + z^2 + 2x - 2y = 2 = 4x^2 + z^2 + 4 + 4zx - 8x - 4z$

$\Rightarrow$ $3x^2 - y^2 + 4zx - 10x + 2y - 4z + 6 = 0$

Example : Prove that the lines drawn from the origin so as to touch the sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ lies on the cone

$d(x^2 + y^2 + z^2) = (ux + vy + wz)^2$

Solution : The given equation of the sphere is

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Since the vertex is (0, 0, 0), so that

$S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$

and $s_1 = x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$

Since $x_1 = 0, y_1 = 0, z_1 = 0$

Therefore, $x_1 = 0 + 0 + 0 + 0 + 0 + 0 + d = d$

and $T = xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d$

$= ux + vy + wz + d$

Therefore, the locus of the line through the origin and touching the given sphere, the enveloping cone is

$ss_1 = T_2$

$\Rightarrow$ $d(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d) = (ux + vy + wz + d)^2$

$\Rightarrow$ $d(x^2 + y^2 + z^2) + (ux + vy + wz + d) + d^2 = (ux + vy + wz)^2$
$+ 2d(ux + vy + wz) + d2$

$\Rightarrow$ $d(x^2 + y^2 + z^2) = (ux + vy + wz)^2$.

5.3.6 RIGHT CIRCULAR CONE

A right circular cone is a surface generated by a straight line which passed through a fixed point of another fixed straight line and makes a constant angle with that as well. The fixed line is called the axis of the cone and the angle made by the generator with the axis is called the semi vertical angle.

Note : The section of the right circular cone by a plane perpendicular to the axis is a circle.

5.3.7 EQUATION OF RIGHT CIRCULAR CONE

Obtain the equation of the right circular cone whose vertex is the origin, axis, the z-axis and semi vertical angle α is

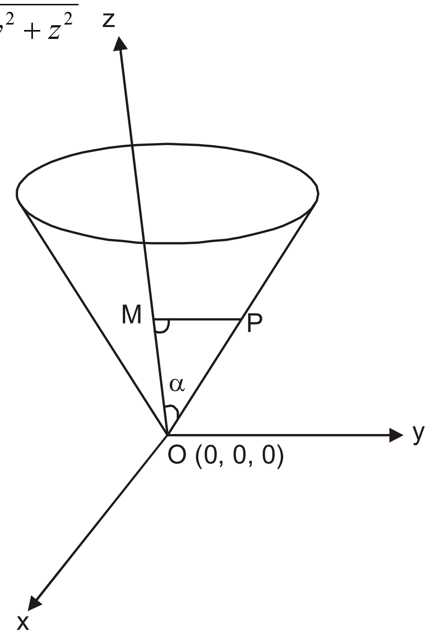
$x^2 + y^2 = z^2 + \tan^2\alpha$.

Solution : Let P(x, y, z) be any point on the cone. Draw PM $\perp$ on the z-axis such that <POM = α. In rt. <d ΔOPM

$OM = OP \cos\alpha$ ---- (1)

But OM = z

and $\sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{x^2+y^2+z^2}$



Putting these values of OP and OM in eq. (1) we get

$$z = \sqrt{x^2+y^2+z^2}\cos\alpha$$

$$\Rightarrow \quad z = \left(x^2+y^2+z^2\right)\cos^2\alpha$$

$$\Rightarrow \quad z^2\sec^2\alpha = x^2+y^2+z^2$$

$$\Rightarrow \quad x^2+y^2+z^2 = z^2(\sec^2\alpha - 1) = z^2 + \tan^2\alpha$$

This is the requried equation of the right circular cone.

5.3.7 FIND THE EQUATION TO THE RIGHT CIRCULAR CONE WHOSE VERTEX IS THE POINT (α, β, γ), THE AXIS IS THE LINE

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

Solution : Let P(x, y, z) be any point on the cone, then the direction ratios of the generator which is the joint of the points (α, β, γ) and (x, y, z) are $x-\alpha,\ y-\beta, z-\gamma$ and these of the axis are l, m, n

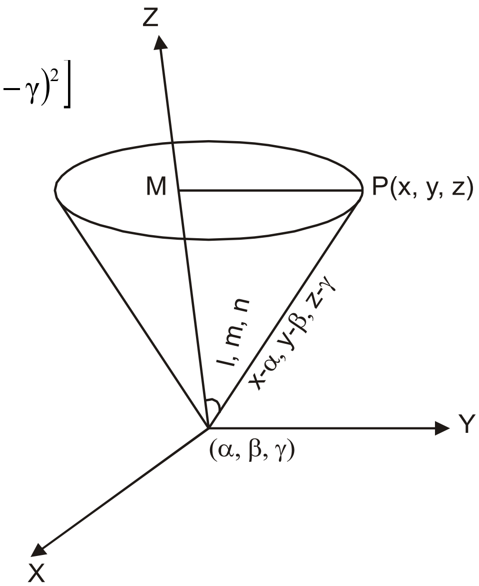
$$\therefore \quad \cos\theta = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

Squaring, we get

$$\cos^2\theta\left(l^2+m^2+n^2\right)\left[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right]$$

$$=l(x-\alpha)+m(y-\beta)+n(z-\gamma)$$

This is the equation of the right circular cone



Example : Prove that

$x^2 + y^2 + z^2 - 2x + 4y + 6z + 6 = 0$ represents a right cicular cone whose vertex is the point (1, 2, -3) and whose axis is parallel to oy and whose semi vertical angle is 45°.

Solution : We have $\alpha = 1, \beta = 2, \gamma = 3$

$l = 0$, $m = 1$, $n = 0$ and θ = 45°

Therefore, the equation of the right circular cone becomes

$(y - 2)^2 = [(x-1)^2 + (y - 2)^2 + (z + 3)2]^2 \cos^2 45°$

or $(x - 2)^2 + (y - 2)^2 + (z + 3)^2 = 2(y - 2)^2$

or $(x - 1)^2 + (y - 2)^2 + 2(y - 2)^2 + (z + 3)^2 = 0$

or $(x - 1)^2 - (y - 2)^2 + (z + 3)^2 = 0$

$\Rightarrow$ $x^2 - 2x + 1 - y^2 - 4 + 4y + z^2 + 9 + 6z = 0$

$\Rightarrow$ $x^2 - y^2 + z^2 - 2x + 4y + 6z + 6 = 0$

Which is the required equation.

Example : Show that the equation to the right circular cone whose vertex is at the origin, whose axis has direction cosumes l, m, n whose semi vertical angle is θ is

$$\Sigma(yn - zn)^2 = \sin^2\theta(x^2 + y^2 + z^2).$$

Solution : We know that the equation of the right circular cone whose vertex is at the origin and semi vertical angle θ is

$$(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2)\cos^2\theta$$

$\Rightarrow$ $(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2) - (1 - \sin^2\theta)$

$\Rightarrow$ $(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2) - (l^2 + m^2 + n^2)(x^2 + y^2 + z^2)\sin^2\theta$

$\Rightarrow$ $(l^2 + m^2 + n^2)(x^2 + y^2 + z^2) - (lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2)\sin^2\theta$

$$(m^2 + n^2)x^2 + (n^2 + l^2)y^2 + (l^2 + m^2)z^2 - 2mnyz - 2nlzx - 2hxy$$
$$= (l^2 + m^2 + n^2)(x^2 + y^2 + z^2)\sin^2\theta$$

or $(ny - mz)^2 + (lz + nz)^2 + (mx - ly)^2 = (x^2 + y^2 + z^2)\sin^2\theta$

$$\left[\therefore l^2 + m^2 + n^2 = 1\right]$$

Substituting these values in the equation of the right circular cone, we have

$$(y - 2)^2 = [(x - 1)^2 + (y - 2)^2 + (z + 3)^2]\cos^2 45°$$

$\Rightarrow$ $[(x - 1)^2 + (y - 2)^2 + (z + 3)^2] = 2(y - 2)^2$

$\Rightarrow$ $(x - 1)^2 + (y - 2)^2 + (z + 3)^2 - 2(y - 2)^2 = 0$

$\Rightarrow$ $(x - 1)^2 - (y - 2)^2 + (z + 3)^2 = 0$

$\Rightarrow$ $(x^2 - 2x + 1)^2 - (y^2 - 2y + 4)^2 + (z^2 + 2z + 9) = 0$

$\Rightarrow$ $x^2 - y^2 + z^2 - 2x + 2y + 2z + 6 = 0.$

EXERCISES

1. Find the equation of the right circular cone which passes through the point (1, 1, 2) and has vertex at the origin and axis is the line

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}.$$

2. Find the equation to the right circular cone whose vertex is P(2, -3 5), axis PQ which makes equal angles with the axes and which pass through the point a (1, -2, 3)

3. Show that $33x^2 + 13y^2 - 95y^2 - 144yz - 96zx - 48xy = 0$ represents a right circular cone whose axis is the line $3x = 2y = z$. Find the semi vertical angle.

ANSWERS

1. $4x^2 + 40y^2 + 19z^2 - 72yz + 36zx - 48xy = 0$

2. $x^2 + y^2 + z^2 + 6(xy + yz + zx) - 16x - 36y - 4z - 28 = 0$

3. $\alpha = 60°$

****

5.4.1 CYLINDER

It is a surface generated by a straight line which moves parallel to a fixed straight line (called axis) and either intersects given curve (called the guiding curve) or touches a given surface.

Note : Any straight line on the cylinder is known as its generators are parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \text{........ (1)}$$

and intersect the conic

$ax^2 + by^2 + 2hyx + 2gx + 2fy + c = 0, z = 0$ .... (2)

Solution : Let (α, β, γ) be any point on the cylinder. Equation of the generator through the point (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{........ (1)}$$

This generator meets the plane z = 0 in the point given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \text{ i.e., } \left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n},0\right)$$

This point will lie on $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0$

$$\Rightarrow \quad a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2+2g\left(\alpha-\frac{l\gamma}{n}\right)$$

$$+2f\left(\beta-\frac{m\gamma}{n}\right)^2+c=0$$

$$\Rightarrow \quad a(n\alpha-l\gamma)^2+2h(n\alpha-l\gamma)(n\beta-m\gamma)+b(n\beta-m\gamma)^2+2g(n\alpha-\gamma l)$$

$$+2f(n\beta-m\gamma)+c=0$$

Hence the locus of the point (α,β,γ) is

$a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(ny - mz)^2 + 2g(nx - lz) + 2f(ny - mz) + c = 0.$

Example : Find the equation of cylinder with the guiding curve given by

$x^2 + 2y^2 = 1$, $z = 0$ and generator parallel $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

Solution : Let (α,β,γ) be any point on the cylinder.

Therefore the equation of generator through the point (α,β,γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{........ (1)}$$

Since the generator passes through $x^2 + 2y^2 = 1$, $z = 0$, we get

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\Rightarrow x=\alpha-\frac{\gamma}{3},\ y=\beta-\frac{2\gamma}{3}$$

Therefore, we have

$$\left(\alpha-\frac{\gamma}{3}\right)^2,2\left(\beta-\frac{2\gamma}{3}\right)^2=1$$

Hence the locus of the point (α, β, γ) is

$$\left(x-\frac{z}{3}\right)^2+2\left(y-\frac{2z}{3}\right)^2=1$$

This is the required equation of cylinder

Example : Find the equation of cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and guiding curve is the ellipse

$x^2 + 2y^2 = 1, z = 3$

Solution : Let (α, β, γ) be any point on the cylinder. Equation of the generator through the point (α, β, γ) are

$$\frac{x-\alpha}{1}=\frac{y-\beta}{-2}=\frac{z-\gamma}{3} \qquad \text{........ (1)}$$

Since the generator passes through $x^2 + 2y^2 = 1, z = 3$, we get

$$\frac{x-\alpha}{1}=\frac{y-\beta}{-2}=\frac{3-\gamma}{3}$$

This generator meets the plane z = 3 in the point

$$\left(\frac{3(\alpha+1)-\gamma}{3},\frac{3(\beta-2)+2\gamma}{3},3\right)$$

Therefore, we have

$$\left(\frac{3(\alpha+1)-\gamma}{3}\right)^2+2\left(\frac{3(\beta-2)+2\gamma}{3}\right)^2=1$$

$\Rightarrow$ $$3(\alpha^2+2\beta^2+\gamma^2)+8\beta\gamma+2\gamma\alpha+6\alpha-24\beta-18\gamma+24=0$$

Hence the locus of the point (α, β, γ) is

$3(x^2 + 2y^2 + z^2) = 8yz + 2zx + 6x - 24y - 18z + 24 = 0$

This is the required equation of cylinder.

EXERCISES

1. Find the equation of the cylinder with the guiding curve given by

   $3x^2 + 4y^2 = 12$, $z = 0$ and the generator parallel to $\frac{x}{2} = \frac{y}{5} = \frac{z}{3}$.

2. Find the equation of cylinder whose generating parallel to z - axis and which passes through the curve of intersection of the surface represented by $x^2 + y^2 + 2z^2 = 12$ and $x + y + z = 1$.

3. Find the equation of cylinder whose generators are parallel to z - axis and which passes through the curve of intersection of $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$.

ANSWERS

1. $x^2 + y^2 - x^2 + xy = 12$

2. $3x^2 + 4xy + 3y^2 - 4x - 4y - 10 = 0$

3. $2y^2 + 3z^2 - 2yz + 2y - 2z - 11 = 0$

5.4 ENVELOPING CYLINDER

Definition : The focus of the tangents to a sphere which are parallel to a given line is called the enveloping cylinder.

5.4 TO FIND THE EQUATON OF THE ENVELOPING CYLINDER OF SPHERE $x^2+y^2+z^2 = a^2$, WHOSE GENERATORS ARE PARALLEL TO THE LINE $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

Solution : The sphere is

$$x^2 + y^2 + z^2 = a^2 \qquad \text{...... (1)}$$

And the given line is $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ ...... (2)

Let (α, β, γ) be any ponit on the cylinder and the tangent which is parallel to line (2) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{...... (3)}$$

Therefore, any point on (3) is $(\alpha+lr, \beta+mr, \gamma+nr)$. This point will be one (1) if

$$(\alpha - lr)^2 + (\beta - mr)^2 + (\gamma - nr)^2 = a^2$$

$$\Rightarrow \qquad r^2(l^2+m^2+n^2)+2r(\alpha r+\beta m+\gamma n)+\alpha^2+\beta^2+\gamma^2-a^2=0 \qquad \text{--- (4)}$$

Which is a quadratic equation in r giving two values of r corresponding to which there are two points common to the sphere (1) and the line (3).

Now if the line (3) be a tangent to (1), both the values of r given by (4) must be equal.

Therefore, we must have

$$(\alpha l+\beta n+\gamma n)^2 = (l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2)$$

Hence the locus of the point (α, β, γ) is

$$(\alpha l + \beta n + \gamma n)^2 = (l^2 + m^2 + n^2)(\alpha^2 + \beta^2 + \gamma^2 - a^2)$$

This is the required equation of enveloping cylinder.

Example : Find the equation of the envelopng cylinder of this sphere $x^2 + y^2 + z^2 + 4y = 1$, having its generator parallel to the line

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

Solution : The given sphere is

$$x^2 + y^2 + z^2 + 4y - 1 = 0 \qquad \text{...... (1)}$$

and the given line is $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$ ...... (2)

Let (α, β, γ) be any point on a tangent which is parallel. This lies on (1) if

$$(\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2 - 2(\alpha + r) + 4(\beta + r) - 1 = 0$$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 + 2\alpha r + 2\beta r + 3r^2 - 2\alpha - 2r + 4\beta + 4r - 1 = 0$$

$$\Rightarrow \quad (\alpha^2 + \beta^2 + \gamma^2 - 2r + 4\beta - 1) + 3r^2 + 2r(\alpha + \beta + \gamma + 1) = 0$$

$$\Rightarrow \quad 3r^2 + 2r(\alpha + \beta + \gamma + 1)r + (\alpha^2 + \beta^2 + \gamma^2 - 2r + 4\beta - 1) = 0 \qquad \text{...... (3)}$$

which is quadratic in r giving two values of r. The line touches the sphere if (3) has equal roots. Therefore, we have

$$4(\alpha + \beta + \gamma + 1)^2 - 12(\alpha^2 + \beta^2 + \gamma^2 - 2r + 4\beta - 1) = 0$$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \alpha\gamma - 4\gamma - 5\beta - \gamma - 2 = 0$$

Hence the locus of (α, β, γ) is

$x^2 + y^2 + z^2 - xy - yz - zx - 4x - 5y - z - 2 = 0$

This is the required equation of the enveloping cylinder.

EXERCISES

1. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = a^2$ whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

2. Find the equation of the cylinder of whose guiding curve is $x^2+y^2+z^2=9$, $x - y + z = 3$.

****

5.5.1 RIGHT CIRCULAR CYLINDER

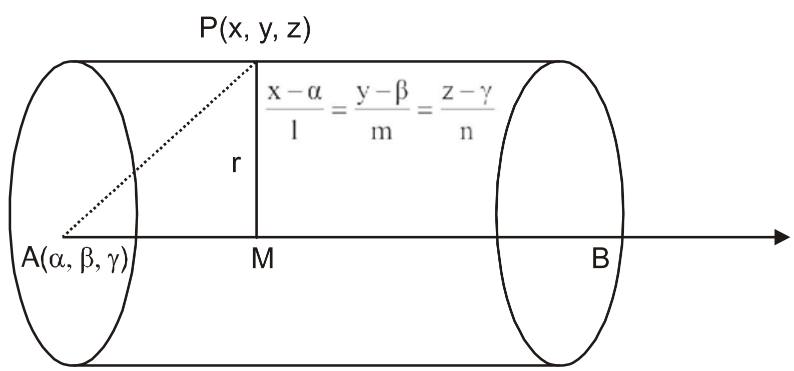
Definition : Right circuar cylinder is the surface generatewd by a straight line which is parallel to a fixed line and is at a constant distance from it. The fixed line is called the axis of the cylinder and constant distanc is called the radius of the cylinder.

5.5.2 EQUATION OF RIGHT CIRCULAR CYLINDER

To find the equation of a righ circular cylinder whose radius is r and the axis of he line

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n} \qquad \text{---- (1)}$$

Let (α, β, γ) be any point on the line (1)



Therefore, in rt $\angle$d ΔAPM

$AP^2 = AM^2 + PM^2$ .... (2)

But $AP^2 = (x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2$

am = projection of ap and ab = $\dfrac{(x-\alpha)l + (y-\beta)m + (z-\gamma)n}{\sqrt{l^2 + m^2 + n^2}}$

and PM = r

From (2), we hav e

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = \frac{[(x-\alpha)l + (y-\beta)m + (z-\gamma)n]^2}{l^2 + m^2 + n} + r^2$$

Hence the required equation of the cylinder is

$$l^2 + m^2 + n^2\left[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 - r^2\right]$$

$$= [(x-\alpha)l + (y-\beta)m + (z-\gamma)n]^2$$

Example : The axis of a right circular cylinder of radius 2 has equation

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$$

Show that its equation is

$10x^2 + 5y^2 + 13z^2 - 12xy - 6yz - 4zx - 8x + 30y - 74z + 59 = 0$

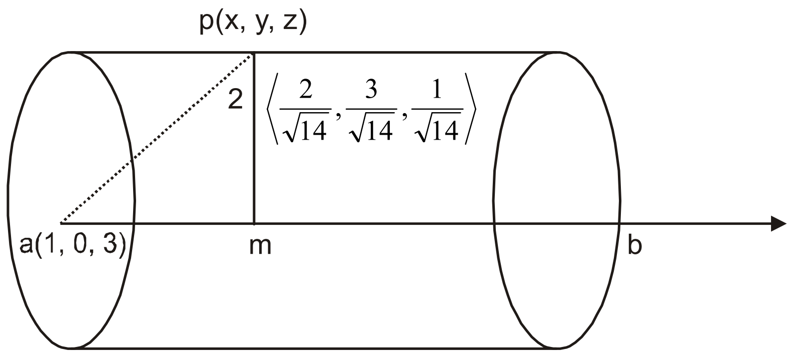
Solution : The given axis ab is $\dfrac{x-1}{2} = \dfrac{y-0}{3} = \dfrac{z-3}{1}$

Its direction cosmies are porportional to <2, 3, 1>

Therefore, direction cosine's are $\left\langle \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}, \dfrac{1}{\sqrt{14}} \right\rangle$.

Let p(x, y, z) be any point on the cylinder.

Draw $pm \perp ab$ and a(1, 0, 3) is a point on the axis.



In rt. angled ΔAPM

$ap^2 = am^2 + pm^2$ ---- (1)

But $AP^2 = (x - 1)^2 + (y - 0)^2 + (z - 3)^2$

$$\Rightarrow \quad ap = \sqrt{(x-1)^2 + (y-0)^2 + (z-3)^2}$$

$$= (x-1)\frac{2}{\sqrt{14}} + y\frac{3}{\sqrt{14}} + (z-3)\frac{1}{\sqrt{14}} = \frac{2x+3y+z-5}{\sqrt{14}}$$

and PM = 2

From (1), we have

$$(x-1)^2 + y^2 + (z-3)^2 = \frac{(2x+3y+z-5)^2}{14} + 14$$

$\Rightarrow \quad 14(x^2 + y^2 + z^2 - 2x - 2z + 2) = (2x + 3y + z)^2 - (10(2x + 3y + Z) + 25 + 56$

$\Rightarrow \quad 10x^2 + 5y^2 + 13z^2 - 12xy - 6yz - 4zx - 8x + 30y - 74z + 59 = 0$

Example : Find the equation of the right circular cyinder whose axis is

$$\frac{1}{2}x = \frac{1}{3}y = \frac{1}{6}z$$

and radius 5.

Solution : Let (x, y, z) by any point on the cylinder. Then the length of the perpendicular from $P(x_1, y_1, z_1)$ to the given line $\frac{x}{2} = \frac{y}{3} = \frac{z}{6}$ must be equal to the radius 5, to we get

$$5^2[2^2 + 3^2 + 6^2] = (6y_1 - 3z_1)^2 + (2z_1 - 6x_1)^2 + (3x_1 - 2y_1)^2$$

$\Rightarrow \quad 25(49) = 36y_1^2 + 9z_1^2 - 36y_1z_1 + 4z_1^2 + 36x_1^2 - 24z_1x_1 + 9x_1^2 + 4z_1^2 - 12x_1y_1$

$\Rightarrow \quad 45x_1^2 + 40y_1^2 + 13z - 12x_1y_1 - 36y_1z_1 - 24z_1x_1 - 1225 = 0$

$\therefore$ The locus of $P(x_1, y_1, z_1)$

i.e., the required equation is

$$45x^2 + 40y^2 + 13z^2 - 12xy - 36yz - 24zx - 1225 = 0$$

Example : Find the equation to the right circular cylinder whose guiding circle is

$$x^2 + y^2 + z^2 = 9,\ x - y + z = 3.$$

Solution : The sphere $x^2 + y^2 + z^2 = 9$ .... (1)

and the plane is $x - y + z = 3$ .... (2)

The centre of sphere is O (0, 0, 0) and its radius is ob = 3

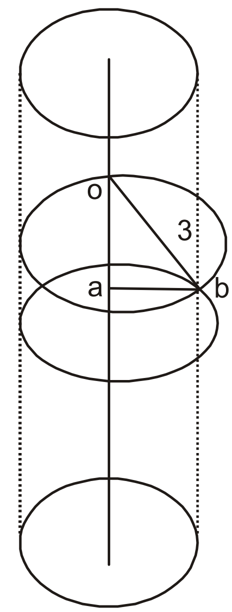
oa = $\perp$ distance of O (0, 0, 0) from the plane (2)

$$= \frac{0 - 0 + 0 - 3}{\sqrt{1+1+1}} = -\sqrt{3} = \sqrt{3} \text{ (numerically)}$$

$\therefore$ ab = radius of circle

$= \sqrt{ob^2 - oa^2} = \sqrt{9-3} = \sqrt{6}$

Again equation of the line through the centre O(0, 0, 0) of the sphere and $\perp$ to the plane (2) are



$\frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$, which is the axis of the cylinder

Let p (x, y, z) by any point on the cylinder. Join OP and draw mp $\perp$ to the axis oa, so that

mp = radius of a cylinder on circle = $\sqrt{6}$

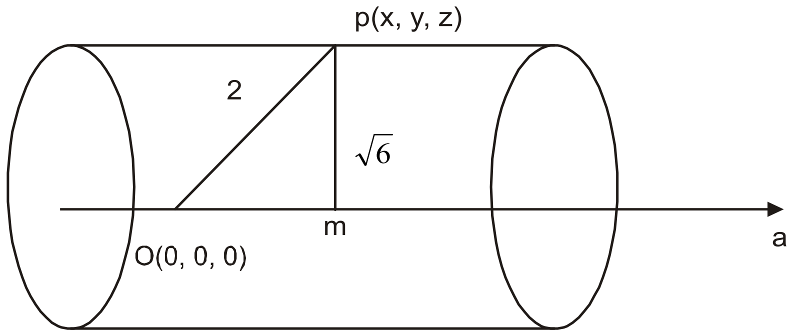
$op^2 = mp^2 + om^2 = 6 + om^2$ --- 3)

Now $op^2 = (x - 0)^2 + (y - 0)^2 + (z - 0)^2 = x^2 + y^2 + z^2$

and om : projectection of op on the line oa whose actual direction cosmies are $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

$= \frac{1}{\sqrt{3}}(x - 0) - \frac{1}{\sqrt{3}}(y - 0) \frac{1}{\sqrt{3}}(z - 0)$

$= \frac{x - y + z}{\sqrt{3}}$ $[\because l(x_2 - x_1) + m(y_2 + y_1) + n(z_2 - z_1)$



From eq (3), we have

$$x^2 + y^2 + z^2 = \left(\frac{x - y + z}{\sqrt{3}}\right)^2 + 6$$

$\Rightarrow \quad 3(x^2 + y^2 + z^2) = (x - y + z)^2 + 18 = x^2 + y^2 + z^2 - 2xy - 2yz + 2zx + 18$

$\Rightarrow \quad 2x^2 + 2y^2 + 2z^2 + 2xy + 2yz - 2zx - 18 = 0$

$\Rightarrow \quad x^2 + y^2 + z^2 + xy + yz - zx - 9 = 0$, which is the required equation of the cylinder

Example : Prove that the equation of the right circular cylinder whose one section is the circle

$x^2 + y^2 + z^2 - x - y - z = 0, x + y + z = 1$

$x^2 + y^2 + z^2 - yz - xy - xy = 1.$

Solution : The direction ratio of the axis of the cylinder which is perpendicular to the plane of the circle given by $x + y + z = 1$ are 1, 1, 1

So, let one of the generators of the cylinder passing through any point $p(\alpha, \beta, \gamma)$ on the cylinder be

$$\frac{x-\alpha}{1}=\frac{y-\beta}{1}=\frac{z-\gamma}{1}$$

Any point on this generator at a distance r from (α,β,γ) is $(\alpha+r,\beta+r,\gamma+r)$. If the point lies on the given circle, we have

$$(\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2-(\alpha+r)-(\beta+r)+(\gamma+r)=0$$

$$=(\alpha+r)+(\beta+r)+(\gamma+r)=1$$

or $(\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2-1=0,(\alpha+\beta+\gamma)+3r=1$

Eliminating r, we get

$$\left[\alpha+\frac{1-(\alpha+\beta+\gamma)}{3}\right]^2+\left[\beta+\frac{1-(\alpha+\beta+\gamma)}{3}\right]^2+\left[\gamma+\frac{1-(\alpha+\beta+\gamma)}{3}\right]^2=1$$

$$(2\alpha-\beta-\gamma+1)^2+(2\beta-\gamma-\alpha+1)^2+(2\gamma-\gamma-\beta+1)^2=9$$

or $6\alpha^2+6\beta^2+6\gamma^2-6\beta\gamma-6\gamma\alpha-6\alpha\beta)=6$

$\therefore$ The required equation of the cylinder is

$x^2+y^2+z^2-yz-zx-xy=1$

EXERCISE

1. Find the equation to the right circular cylinder of radius 2 and whose axis is the line. $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$.

2. Find the equation to the right circular cylinder of radius 3 and whose axis passes through (1, -1, 3) and has direction cosmies proportional <2, -1, 3>.

3. Show that the equation of the right circular cylinder described on the circle through the parents (1, 0, 0), (0, 1, 0) and (0, 0, 1) as the guiding curve is $x^2 + y^2 + z^2 - yz - zx - xy = 1$.

4. Find the equation of the right circular cylinder whose axis is $x + 2y = -z$ and radius 4.

ANSWERS

1. $5x^2 + 8y^2 + 5z^2 - 4xy - 8zx - 4yz + 22x - 16y - 14z = 10$

2. $10x^2 + 13y^2 + 5z^2 + 6yz - 12zx - 4xy + 8x + 10y - 2z - 123 = 0$

3. $5x^2 + 8y^2 + 5z^2 + 4yz + 8zx - 4xy - 144 = 0$

****