B.A. SEM-IV

SYLLABUS

(AS PRESCRIBED BY UNIVERSITY OF JAMMU)

UNIT-I

Aximatic definition of real number as a complete ordered field least up bound property. Sequence and their limits. Uniqueness of limit, Algebraic limit, bounded and monotonic sequence.Cauchy's general principle of convergence and Nested interval property. Examples and exercise based on these topics.

UNIT-II

Infinite series and their examples, positive term series, convergence and divergence of series, tests for convergence : p-test. Comparison test, Ratio test and Rabbe's test. Alternating series and Leibnitz test. Absolute convergence of series. Examples and exercieses based on these topic.

UNIT-III

Definitions and examples of continuous and discontinuous functions and definition of continuity with elementary illustrations, continuity uniform continuity on the closed and bounded intervals, every continuous functions attains its bounds on the closed intervals.

Differentiable functions and examples. Rolle's Therom ..... Theorem, Lagrange's Theorem, Cauchy's Theorem and Taylor's Theorem with Lagrange's form of remainder, Taylor's Series ...... some functions. Example and exercieses Based on those

UNIT-IV

De Moivre's Theorem and its application in finding roots of Complex numbers and expressions of power of sine and cosines in terms of multiples of Q and vice-versa. Functions of Complex Variables, Exponential Functions and Logarithimic Functions. Examples, problems and exercises based on these topics.

UNIT-V

Circular, Hyperbolic and Inverse Circular functions of Complex Variable and their properties. Summation of n terms trigonometric series, hyperbolic and logarithmic function and C + iS method. Examples, problems and exercise based on these topics.

CONTENTS

THE REAL NUMBER SYSTEM

1.1. Introduction: In this lesson the concept of numbers such as natural numbers, whole numbers, integers, real numbers etc.are given.

1.2. Objectives: Objective of studying this lesson is to give the idea how to construct real numbers.

1.3. INTRODUCTION

An understanding of the real number system is basic to a thorough understanding of analysis. There are several ways in which the study could be presented. Our way w a start with the numbers, 1, 2, 3,.... (the "counting", numbers, or "natural numbers"), to the set of integers, and then construct the larger system of rational numbers; finally number system could be constructed from the rationals as given in the following definition:

Definitions. 1.3.1 :

(i) The set of natural numbers is denoted by N and defined as

$$N = \{1, 2, 3 \ldots\}.$$

(ii) The set of integers is denoted by I or Z and defined as :

$$Z \text{ or } I = \{.......... -3, -2, -1, 0, 1, 2, 3, ...\}$$

(iii) The set of rational numbers is denoted by Q and defined as :

$$Q = \left\{\frac{p}{q} : q \neq 0 \text{ and } p, q \in Z\right\}$$

(iv) The set of irrational numbers is denoted by I_r, and defined as :

$I_r = \{x \mid x \notin Q\}$, *i.e.* I_r, consists of all those numbers which are not rational.

(v) The set of real numbers is denoted by R and defined as:

$R = Q \cup I_r$, *i.e.*, the collection of all rational and irrational numbers is called the set of real numbers.

The two fundamental operations in the real number system are **addition** and **multiplication.** They are often called **binary operations** because they serve to combine two elements (numbers) in prescribed ways. The familiar operations subtraction and division are defined in addition and multiplication respectively. We shall start with eleven axioms, five of which (A_1 through A_5) describe addition, a similar five (M_1 through M_5) which describe multiplication, and one (labeled D) which interrelates the two operations in a particular way. We have:

A_1. Every pair of numbers a and b in R have a unique sum a + b, which is also in R. *(Closure law for addition).*

A_2. For a and b in R, $a + b = b + a$, *(Commutative law for addition).*

A_3. For a, b and c in R, $a + (b + c) = (a + b) + c$.

(Associative law for addition)

A_4. There is a number 0 in R such that for each a in R, $a + 0 = a = 0 + a$.

(Existence of an additive identity)

A_5. For every $a \in R$, there exists a number $-a$ in R such that $a + (-a) = 0 = (-a) + a$. *(Existence of additive inverse).*

The difference between a and b is defined as $a + (-b)$ and the indicated operation iscalled **subtraction**. Often $a - b$ is used as an abbreviation for $a + (-b)$. The symbol $-b$ should be called "the additive inverse of b" or simply the "negative of b".

M_1. Every pair of numbers a and b in R have unique product ab, which is also in R. *(Closure law for multiplication)*

M_2. For a and b in R, $ab = ba$. *(Commutative law for multiplication.)*

M_3. For a, b and c in R $a(bc) = (ab)c$. *(Associative law for multiplication)*

M_4. There exists a number I in R, where $I \neq 0$, such that for each a in R, $a.1 = a = 1.a$. *(Existence of multiplicative identity)*

M_5. For every $a \neq 0$ in R there exists a number, denoted by a^{-1}in R, such that

$a.a^{-1} = 1 = a^{-1}.a$. *(Existence of multiplicative inverse).*

The quotient of a and b, $(b \neq 0)$, is defined as $a.b^{-1}$, or equivalently, $b^{-1}.a$ and the indicated operation is called **division**. The common way of denoting the quotient is $\frac{a}{b}$.

D. For a, b and c in R $a\,(b + c) = ab = ac$.

[*Distributive law of multiplication over addition*]

These eleven axioms are called the field axioms of real number system.

Definition 1.3.2. Any set F with two binary operations '+' and '.' is said to be a field if it satisfies the laws $A_1 - A_5$, $M_1 - M_5$ and D.

For example. The set Q of all rational numbers is a field under the usual operations of addition and multiplication.

Example. The set of N of natural numbers is not a field .(because there is no additive identity element in N).

Example. The set Z of integers is not a field under the usual addition and multiplication compositor (why).

The real number system requires other axioms in addition to those for its complete description,but before presenting further axioms we shall prove some theorems concerning based only upon the axioms already stated.

Theorem 1.3.3. The cancellation law for addition :

$$b + a + c + a \text{ implies that } b = c, \text{ for all } a, b, c \in \text{R}.$$

Proof : $b + a = c + a$

$\Rightarrow \quad (b+a)+(-a)=(c+a)+(-a)$

$\Rightarrow \quad b+(a+(-a))=c+(a+(-a))$

$\Rightarrow \quad b + 0 = c + 0$

$\Rightarrow \quad b = c$

Theorem1.3.4. (Cancellation law of multiplication)

If $x, y, z \in$ R such that $xy = xz$ and $x \neq 0$, then $y = z$.

Proof : As $x \neq 0$, so x^{-1} exists. Thus $xy = xz$

$\Rightarrow \quad x^{-1}(xy) = x^{-1}(xz)$

or $\quad (x^{-1}x)\,y = (x^{-1}x)\,z$

$\quad 1.y = 1.z$

or $\quad y = z$

Theorem 1.3.5. There can exists at the most one identity element:

(i) for addition (ii) for multiplication

In**R**.

Proof: (i) If possible, suppose 0 and 0' be two real numbers such that for each $x \in$ R.

$$x + 0 = x,\ x + 0' = x$$

Since $x + 0 = x \ \forall\ x \in$ R, therefore in particular $0' + 0 = 0'$ ...(i)

Again, since $x + 0' = x$, $\forall\, x \in$ R therefore, in particular

$$0 + 0' = 0 \qquad ...(ii)$$

From (i), (ii) and using–commutative law under addition, we have

$$0' = 0' + 0 = 0 + 0' = 0$$

Hence additive identity is unique in R.

(ii) Similar to (i) part.

It is often good idea to restate a theorem into the form of an implication in order to make the proof move understandable. Be sure that the restatement is equivalent to the original theorem.

1.4. EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

Each of the following statements about real numbers is a theorem based on the eleven field axioms. Prove each one in a manner similar to the proof of the proceeding theorems.

1. The additive identity is unique.

   [**Hint :** Consider the restatement "if an element b has the property a+b = a for all real numbers a, then b = 0];

2. The additive inverse of the additive inverse of a real number–b is b itself , i.e.—(—b)=b.

3. The negative of zero is zero itself, i.e. —0 = 0.

4. The cancellation law for multiplication holds: i.e. ba = ca, and $a \neq 0$ imply b=c.

5. The multiplicative inverse of a non zero number is unique.

6. The multiplicative identity is unique.

7. $I^{-1}=1$

8. $(c - b) + (b - a) = c - a.$

9. The additive inverse of $a + b$ is $- a - b$, *i.e.* $-(a + b) = -a - b.$

[Note that $-a - b$ is the abbreviation for $(-a) + (-b)$].

10. $(c + a) - (c + b) = a - b.$

1.5. THE AXIOMS OF ORDER

In addition to the field axioms, the real numbers have an order relation, “>; <“ which is based on the following axioms.

O_1: Given any two real numbers a, b one and only one of the following holds:

$a > b,\ a = b,\ b > a$ *[Law of Trichotomy]*

O_2 : For any real numbers *a, b, c* if $a > b$, $b > c$, then $a > c$. *[Transitivity]*

Q_3 : For all real numbers *a, b* and c, $a > b \Rightarrow a + c > b + c.$ *[Monotone law of addition]*

O_4 : For all real numbers *a, b* and *c*, $a > b$ and $c > 0 \Rightarrow ac > bc.$

The field of real numbers together with O_1 through O_4 is called ordered field so we have following definition:

Definition. Any field (F, +, .) which has the properties O_1, O_2, O_3 and O_4 is called an ordered field.

Example. Q the set of rational numbers is an ordered field.

Remark : Imposibility of Ordering the Complex numbers. The notion of linear ordering < does not apply to complex numbers. If possible, suppose we can define an order relation < satisfying axioms Q_1 to Q_5 of 1.5 . Then since $i \neq 0$, we have either $i > 0$ of $i < 0$ by axioms Q_1. Assume $i > 0$. Then taking $a = b = i$ in axiom Q_4. we get $i.i > 0$ *i.e.* $-1 > 0$. Adding 1 to both sides (axiom Q_3),we get 0 > I. Again applying Axiom Q_4 to $-1 > 0$ and $-I > 0$, we see that $(-1).(-1) > 0$ or $1 > 0$. Thus we have both 0 > 1 and 1 > 0 which contradicts axiom Q_1.Similarly we cannot have 1 < 0.Hence complex numbers cannot be ordered in such a way that axioms $Q_1$to Q_5 are satisfied.

Since $| z |$, R (z) and I (z) are real numbers, the statements like $| z_1 | < | z_2 |$, R (z_1) < R (z_2) and I (z_1) > F (z_2) are meaningful. Also since $| z |^2 = R^2 (z)$, it is easy to see that $| z |$ | R(z) | R(z) and $| z |$ I(z) I(z).

1.6. ABSOLUTE VALUE

Definition1.6.1. The absolute value of a real number x is written as $| x |$, is defined by

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

It is clear that $|x|$ is never negative *i.e.* $|x| \geq 0$.

Thus we always have

$$|x| \geq 0$$

Geometrical meaning of Absolute value of x is the distance of point P from origin

i.e. If P is the position of point corresponding to real no. x, then distance from origin O to P is $|x|$ or OP = $|x|$.

Note : Also by definition

$$|-x| = |x|$$

Some theorems which are immediate consequences of the definitions will now follows :

Theorem 1.6.2. $|x| = \max(x, -x)$

Now $\qquad |x| = x \geq -x$ if $x \geq 0$

Also $\qquad |x| = -x > x$, if $x < 0$

Thus in either case |x|is greater of the two numbers, x –x, i.e.,|x|= max(x,–x)

Corollary 1.6.3 $\quad |-x| = \max(-x, -(-x))$

$$= \max(-x, x) = |x|$$

$$|-x| = |x|.$$

Corollary 1.6.4. $\quad |x| = \max(x, -x) \geq x$

$$|x| \geq x,$$

Theorem 1.6.5. $\quad -|x| = \min(x, -x)$

Now $\qquad -|x| = x < x$, if $x > 0$

Also $\qquad -|x| = (-x)\, x < -x$, if $x < 0$

Thus in either case –lx| is smaller of the two numbers x and $-x$,

i.e. $\qquad -|x| = \min(x, -x)$

Corollary 1.6.6. $\quad -|x| = \min(x, -x) \leq x$

$$-|x| \leq x$$

Theorem 1.6.7. If $x,\ y \in \mathbf{R}$, then

(i) $|x|^2 = x^2 = |-x|^2$

(ii) $|xy| = |x| . |y|$

(iii) $\left|\frac{x}{y}\right| = \frac{|x|}{|\mathrm{y}|}$, provided $y \neq 0$

Proof : (*i*) For $x \geq 0, |x| = x \quad \Rightarrow \quad |x|^2 = x^2$

For $x < 0,\ |x| = -x \quad \Rightarrow \quad |x|^2 = (-x)^2 = x^2$

Thus in either case $|x|^2 = x^2$

Similarly, $|-x|^2 = (-x)^2 = x^2$

Hence, $|x|^2 = x^2 = |-x|^2$

(*ii*) $|xy|^2 = (xy)^2 = x^2 y^2 = |x|^2 .|y|^2 = (|x|.|y|)^2$

$\therefore \quad |xy| = \pm |x|.|y|$

But since $|xy|$ and $|x| . |y|$ are both non negative, we take only the positive sign.

$\therefore \quad |xy| = |x|.|y|$

(*iii*) $\left|\frac{x}{y}\right|^2 = \left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2} = \left(\frac{|x|^2}{|y|}\right)$ but since $\left|\frac{x}{y}\right|$ and $\left|\frac{x}{y}\right|$ are both non–negative, therefore taking positive square root of both sides, we have

$$\left|\frac{x}{y}\right| = \left|\frac{x}{y}\right|, \text{ when } y \neq 0.$$

Theorem 1.6.8. Triangle inequalities. For all real numbers *x, y,* show that

(i) $|x + y| \leq |x| + |\mathrm{y}|$ and

(ii) $|x - y| \geq |x| - |\mathrm{y}|$

(i) $|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2xy$

$$\leq |x|^2 + |y|^2 + 2|x|.|y| \qquad [\because \ xy \leq |xy| = |x|.|y|]$$

$$= (|x| + |y|)^2$$

Since | $x + y$ | and | x | + | y | are both non–negative, therefore, taking roots on both sides, we have

$$|x + y| \leq |x| + |y|$$

(ii) $|x - y|^2 = (x - y)^2 = x^2 + y^2 - 2xy$

$$\geq |x|^2 + |y^2| - 2|x||y| \qquad [\because -(xy) \geq -|xy| = -|x| - |y|]$$

$$= (|x| - |y|^2) = ||x| - |y||^2$$

Since | $x - y$ | and || x | – | y || are both non–negative, therefore taking the positive square root of both sides, we have

$$|x - y| \geq |x| - |y|$$

EXERCISE

1. | x | = 0 if $x = 0$
2. | $x - y$ | = 0 if $x = y$
3. | $x + y + z$ | | y | + | z |.
4. If | $x - a$ | then $a - x < a + \in$ and $x - \in < a < x + \in$.
5. If x, y, a are reals such that | $x - a$ | $< \in$ and | $y - a$ | $< \in$. Then | $x - y$ |$<$ $2\in$.

1.7. INTERVALS–OPEN AND CLOSED

A subset A of R is called an interval if A contains (i) at least two distinct elements and (ii) every element lies between any two members of A.

Open Interval : If a and b are two real number such that $a < b$ then the set

$$\{x : a < x < b\}$$

insisting of all real numbers between a and b (excluding a and b) is called an open interval d is denoted by $]a, b[$ or (a, b).

Closed Interval : The set $\{x : a \leq x \leq b\}$ insisting of a, b and all real numbers lying between a and b is called a closed interval and denoted by $[a, b]$.

Semi–closed or Semi–open intervals.

$$]a,b] = \{x : a < x \leq b\}$$

$$[a, b[= \{x : a \leq x < b\}$$

The intervals are semi–closed or semi–open. The former is open at a and closed at b while the latter is closed at a and open at b.

Now we define infinite intervals.

(i) The set of all real number x, satisfying $x \geq a$ is denoted by $[a, \infty]$.

Thus $[a, \infty] = \{x \in R : x \geq a\}$

(ii) The set of all real numbers x, satisfying $x > a$, is denoted by (a, ∞).

Thus $(a, \infty) = \{x \in R : x > a\}$

(iii) The set of all real numbers x, satisfying $x > a$, is denoted by $(-\infty, a]$.

Thus, $(-\infty, a) = \{x \in R : x \leq a\}$

(iv) The set of real numbers x, is denoted by $(-\infty, \infty)$. Thus $(-\infty, \infty) = R$.

1.8. COMPLETENESS

The properties of R. listed up till now do not enable us to distinguish between of real numbers and the set Q of rational numbers in as much as both these sets fields.

We now propose to state one more property (and this is last property of R) which will serves to distinguish between the sets R and Q. This property, known as order completeness (or simply completeness) is base on the notion of an upper bound of a set of real numbers.

Definition 1.8.1. Let S denote an non empty set of real numbers. A real number b, where b is not necessarily in S, is called an **upper bound**for S if $x \leq b$ for every x in S.

Example1.8.2. Let S = {1, 3, 5,7}. Then 7 or any number greater than 7 will serve as an upper bounds of S.

Not all subsets of the real numbers have upper bounds.

Example1.8.3. The set S = $\{x/x$ is positive$\}$ does not have an upper bound, because if b is an upperbound for S. then $0 < I > b$, since $I \in S$. Now $b + 1 > b > 0$, so $b + 1$ is positive, and therefore in S and $b + 1$ is greater than the proposed upper bound b. This contradicts the definition of upper bound.

Sets which have an upper bound are said to bounded above.

Definition 1.8.4. A real number c is called the **last upper bound** (abbreviated l.u.b.) or supremum of a set S if.

(i) c is an upper bound for S, and

(ii) for any upper bound b other than c, $b > c$.

Example1.8.5. (i) 7 is the l.u. b of a set S (bounded above) is unique.

(ii) l is the l.u.b for the set {............}.

Solution of Uniqueness. Suppose b and c are upper bounds for S. If $b \neq c$, then $b < c$ or $c < b$ by the law of trichotomy for order relation Consequently b and c both could not be least upper bound.

1.9. BOUNDED AND UNBOUNDED SETS : SUPREMUM, INFIMUM

A subset S of real numbers is said to be **bounded above** if $\exists$ a real number k such that every number of S is less than or equal to k *i.e.* $x \leq k, \forall x \in S$

The number k is called an **upper bound** of S. If no such number k exists, the set is said to be **unbounded above** or **not bounded above.**

The set S is said to be **bounded below** if a real number k such that every member of S is greater than or equal to k, *i.e.* $k < x, \forall x \in S$

The number k is called a **lower bound** of S. If so such number k exists, the set is said to be **unbounded below** or **not bounded below**.

A set said to be **bounded** if it is bounded above as well as below.

It may be seen that if a set has one Upper bound, it has an infinite number of upper bounds. For, if k is an upper bound of a set S then every number greater than k is also an upper bound of S. Thus every set S bounded above determines an infinite set–the set of its upper bounds. Similarly, a set S bounded below in a much as every member of S is a lower bound thereof .Similarly, a set S bounded below determines an infinite set of its lower bound, which is bounded above by the members of S.

A members g of a set S is called the **greatest** member of S if every member of S is less than or equal tog, *i.e.*

(*i*) $g \in S$ (*ii*) $x \leq g, \forall x \in S$

Similarly, a member g of the set of its **smallest** (or the least) member if every member of the set is greater than or equal to g.

Clearly, a set may or may not have the greater or the least member but an upper (lower) bound of the set, if it is a member of the set, is its greater (least) member. A finite set always has the greatest as well as the smallest member.

If the set of all upper bounds of a set S has the smallest members, say M, then M is called the **least upper bound** (*l, u, b*) or the **supremum** of S.

Clearly, the supremum of a set S may or may not exist and in case it exists, it may or may not belong to S. The fact that supremum M is the smallest of all the upper bounds of S may be described by the following two properties.

(i) M is the upper bound of S, *i.e.* $x \leq M, \forall x \in S$

(ii) No number less than M can be the upper bound of *s, i.e.* for any positive number $\in$ however small, $\exists$ a number $y \in S$ such that $y > M - \in$

Again it may be see that *a set cannot have more than one supremum*. For, let it possibleM and M' be two supreme of a set S. so that M and M' are both upper bound of S.

Also M is the *l.u.b.* and M is an upper bound of S.

$$M \leq M$$

Again M' is the *l.u.b.* and M is an upper bound of S.

$$M' \leq M \qquad ...(2)$$

From (1) and (2), it follows that M = M'.

If the set of all tower bounds of a set S has the greatest member, say *m*, then *m* is called the **greatest lower bound** (*g.l.b*) or the **infimum** of S.

Like the supremum, the infimum of a set may or may not exist and it may or may not belong to S. It can be easily shown that a set cannot have more than one infimum.

The infimum *m* of a set S has the following two properties.

(i) mis the lowest bound of S, *i.e.* $m \leq x, \forall x \in S$

(ii) No number greater than *m* can be a lower bound of S, *i.e.* for any positive number, however small, a number $z \in S$ such that $z < m + \in$.

1.9.1. Illustrations :

1. The set **N** of natural numbers is bounded below but not bounded above. 1 is a lower bound.

2. The set **I, Q** and **R** are not bounded.

3. Every finite set of numbers of bounded.

4. The set S_1 of all positive real numbers $S_1 = \{x : x > 0, x \in \mathbf{R}\}$ is not bounded above, but is bounded below. The infimum zero is not a member of the set S_1.

5. The inifinite set $S_2 = \{x : 0 < x < 1, x \in \mathbf{R}\}$ is bounded with supreme 1 and infimum zero, 1 both of which do not belong to S_2.

6. The infinite set $S_3 = \{x : 0 \leq x \leq 1, x \in \mathbf{Q}\}$ is bounded, with supremum 1 and infimum 0 both of which are members of S_3.

7. The set $S_n = \left\{\frac{1}{n} : n \in \mathrm{N}\right\}$ is bounded. The supremum 1 belongs to S_4 while infimum 0 does not.

8. Each of the following intervals is bounded : *[a, b],]a, b], [a, b[,]a, b[.*

1.9.3. COMPLENTENESS IN R.

We have already established that (R, +, ., <) is an ordered field. All these properties of ordered *l* are also satisfied by the system of rational numbers. Thus we can that (Q, –+, .,) is also ordered field. Now we state completeness axiom in **R**, which distinguishes the system of real numbers from the system of real numbers.

Completeness axiom in R. *Even non empty set S of real numbers, that is bounded above, l.u.b* in **R**. It is called least upper bound property of **R.** Due to this least upper bound property, R, the set of reals, is said to be *complete ordered field.*

Now, we shall show that the property of completeness does not hold good in case of ordered ofrational numbers.

Theorem 1.9.4. ***The set of rational numbers is not a complete ordered field.***

Proof. In order to show that the set of rational numbers **Q** is not a complete ordered field, it will coefficient to show that there exists a non empty set **S** of rational i.e. $S \subseteq Q$ which isbounded but its *I.u.b* does not belong to **Q** *i.e.* there is no rational numbers which is *l.u.b* of S.

1.10. EXAMINATION ORIENTED EXERCISE/LESSON END EXERCISE

1. Give several real numbers which serve as upper bounds, and lower bounds, for each of the following sets :

(a) S = {2, 7, –3, 0, 8}

(b) S = ($x/x = n^2 + 2$ where n is a natural number less than 4}

2. Find Supremum of each of the following sets :

(a) $S = \{3, 4\}$ (b) $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$

(c) $S = \left\{\pi + 1, \pi + \frac{1}{3}, \pi + \frac{1}{3}, ...\right\}$

3. Find the infimum of each of the following sets :

(a) $S = \{12, 20\}$ (b) $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$ (c) $S = \left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, ...\right\}$

4. Which of the following sets are bounded below, which are bounded above and which are bounded neither below nor above:

(a) $\{1, 2, 3, 4...\}$ (b) $\{-1, -2, -3, ...\}$

(c) $\left\{2, \frac{3}{2}, \frac{4}{5}, \frac{5}{4}, ..., \frac{n+1}{n}\right\}$ (d) $\left\{2, \frac{3}{2}, \frac{4}{5}, \frac{5}{4}, ..., \frac{n-1}{n}\right\}$

5. Prove that between two rationals, there lies another rational.

6. Prove that $\frac{a+b}{2} \geq \sqrt{ab}$ *i.e.* arithmetic mean $\geq$ Goemetric mean.

Hint : $(a - b)^2 \geq 0$ for any real numbers a and b.

7. For any $a \in R$ if $a > 0$, then $a^{-1} > 0$.

8. (i) Give an example of a set which is not a field.

(ii) Give an example of a field which is not an ordered field.

(iii) Give an example of afield which is not complete, justify you answer.

9. Give an example each of a set :

(i) Which is bounded above but no bounded below.

(ii) bounded below but not bounded above.

(iii) bounded.

(iv) neither bounded above nor bounded below.

10. Find *l.u.b. g.l.b,* if exists.

(i) $\left\{\frac{n+1}{n} : n \in \mathrm{N}\right\}$ (ii) $\left\{\frac{2x+1}{x+5} : |x| < 2\right\}$

(iii) $\left\{\frac{2x+x}{2-x} : x \le x \le 1\right\}$ (iv) $\left\{-\sqrt{1-4x^2} : |x| \le \frac{1}{2}\right\}$

11. Prove following sets are bounded :

(i) $\left\{\frac{(-1)^n n}{n+1} : n \in \mathrm{N}\right\}$ (ii) $\left\{\frac{1}{n^2+1} : n \in \mathrm{N}\right\}$

1.11. SUGGESTED READING

The students are advised to go through following references for details.

1.12. REFERENCES

(1) Real analysis by by J.N. Kapur & H.C. Saxena, S.Chand & Co.

(2) Real analysis by J.N. Sharma & A.R.Vashishtha,Krishna Publication, New Delhi.

(3) Real analysis by Richard R. Goldberg, Oxford & IBH Publication Co. Pvt. Ltd. New Delhi.

(4) A text Book of Real Analysis by Sunil Gupta, Narinder Kumar, Malhotra Brothers Pacca Danga, Jammu.

(5) Real analysis by Jagdish Parsad Mittal & Neeraj Doda, Sharma Publication, Jallandhar.

1.13. MODEL TEST PAPER

Q. 1. (i) Give an example of a set which is not a field.

(ii) Give an example of a field which is not an ordered field.

(iii) Give an example of afield which is not complete, justify you answer.

Q. 2. Give an example each of a set :

(i) Which is bounded above but no bounded below.

(ii) bounded below but not bounded above.

(iii) bounded.

(iv) neither bounded above nor bounded below.

Q. 3. Find *l.u.b. g.l.b,* if exists.

(i) $\left\{\frac{n+1}{n} : n \in \text{N}\right\}$ (ii) $\left\{\frac{2x+1}{x+5} : |x| < 2\right\}$

(iii) $\left\{\frac{2x+x}{2-x} : x \leq x \leq 1\right\}$ (iv) $\left\{-\sqrt{1-4x^2} : |x| \leq \frac{1}{2}\right\}$

Q. 4. Prove following sets are bounded :

(i) $\left\{\frac{(-1)^n n}{n+1} : n \in \text{N}\right\}$ (ii) $\left\{\frac{1}{n^2+1} : n \in \text{N}\right\}$

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SEQUENCE

2.1. Introduction: In this lesson the concept of sequence of numbers is discussed.

2.2. Objectives: Objective of studying this lesson is to explain how a sequence of numbers converges or diverges. Also the properties of these convergent sequence are discussed.

2.3. SEQUENCE

2.3.1. Definition: A sequence is a function whose domain is always the set of natural numbers and range is a subset of R *i.e.* sequence is a function $f : N \rightarrow A, A \subset R$.

Notation : Sequence is generally denoted by $\{f_n\}$ or $\{f(n)\}$

2.3.2. **Range :** Let $f : N \rightarrow A$ be a sequence, then the set $\{f(n) : n \in N\}$ is called a range of a sequence.

2.3.3. Example

$\{1, -1, 1, -1, 1, -1............\}$ with range = $\{1, -1\}$

$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}............\right\}$ with range = $\left\{\frac{1}{n} : n \in N\right\}$

$\{1, 2, 4, 8, 16,.......\}$ with range = $\{2^{n-1} : n \in N\}$

$\{1 + i, 1 + 2i, 1 + 3i, 1 + 4i,.......................\}$ is not a sequence because its range is

$\{1 + ni$ such that $n \in R\} \not\subset R$

2.4. CONVERGENT SEQUENCE

A sequence $\{f_n\}$ is said to converge to a number I (I $\subset$ R), if for $\in > 0$, $\exists a, m \in N$ such that $|f_n - l| < \in, \forall n \geq m$

Symbolically, we write it as

$$\lim_{n\to\infty} f_n = l$$

2.4.1.Example : Show that $\left\{\frac{1}{n}\right\}$ converges to zero

or

Prove that $\frac{1}{n} \to 0$ as $n \to \infty$.

Solution : Let $f_n = \frac{1}{n}, l = 0$. To show $\frac{1}{n} \to 0$ *i.e.* $f_n \to l$. Let $\in > 0$

Consider $|f_n - l| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n}$

$|f_n - l| < \in$, if $\frac{1}{n} < \in$

$|f_n - l| < \in$, if $n > \frac{1}{\in}$

$|f_n - l| < \in$, if $n > m, m = \frac{1}{\in}$

$\Rightarrow \quad f_n \to l \Rightarrow \frac{1}{n} \to 0 \quad$ or $\quad \lim_{n\to\infty} \frac{1}{n} = 0$

2.4.2. Example: Show that $\lim_{n\to\infty} \frac{3n+4}{5n-2} = \frac{3}{5}$

Show $\left\{\left\{\frac{3n+4}{5n-2}\right\}\right\}$ converges to $\frac{3}{5}$

Solution let f_n = , 1 =

We show $\frac{3n+4}{5n-2} \to \frac{3}{5}$ *i.e.* $f_n \to l$

Consider $|f_n - l| = \left|\frac{3n+4}{5n-2} - \frac{3}{5}\right| = \left|\frac{15n+20-15n+6}{5(5n-2)}\right| = \left|\frac{26}{5(5n-2)}\right| = \frac{26}{5(5n-2)}$

$$< \in \text{ if } \frac{26}{5(5n-2)} < \in$$

$$< \in \text{ if } 26 < 5(5n-2)\in$$

$$< \in \text{ if } \frac{26}{5\in} < 5n-2$$

$$< \in \text{ if } \frac{26}{5\in} + 2 < 5n$$

$$< \in \text{ if } \frac{1}{5}\left(\frac{26}{5\in} + 2\right)$$

$$< \in \text{ if } n > \frac{1}{5}\left(\frac{26}{5\in} + 2\right)$$

$$< \in \text{ if } n > m,\ m = \frac{1}{5}\left(\frac{26}{5\in} + 2\right)$$

$\Rightarrow$ $|f_n - l| < \in$ if $n > m$, where $m = \frac{1}{5}\left(\frac{26}{5\in} + 2\right)$

$\Rightarrow$ $f_n \to l$ or $\frac{3n+4}{5n-2} \to \frac{3}{5}$

2.4.3. Example. Show that $\sqrt[n]{n} \to 1$ or $(n)^{\frac{1}{n}} \to 1$ or $\lim_{n\to\infty} (n)^{\frac{1}{n}} = 1$.

Solution. To show $\sqrt[n]{n} \to 1$

Let $f_n = \sqrt[n]{n} - 1, l = 0$

We show $f_n \to 0$ $\Rightarrow$ $1 + f_n = \sqrt[n]{n} = (n)^{\frac{1}{n}}$

Raise power n to both side, we get

$$n = (1 + f_n)^n = 1 + n_{c_1} (1)^{n-1} f_n + n_{c_2} (1)^{n-2} f_n{}^2 + ... + f_n{}^n$$

$$= 1 + nf_n + \frac{n(n-1)}{2.1} f_n{}^2 + ... + f_n{}^n$$

In particular,

$$n > \frac{n(n-1)}{2} f_n^2$$

$$1 > \frac{(n-1)}{2} f_n^2$$

$$\frac{2}{n-1} > f_n^2$$

or $$f_n^2 < \frac{2}{n-1}$$

$$f_n < \pm \sqrt{\frac{2}{n-1}}$$

$$| f_n | < \left| \pm \sqrt{\frac{2}{n-1}} \right| = \sqrt{\frac{2}{n-1}}$$

$$| f_n | < \in \text{ if } \frac{2}{n-1} < \in^2$$

$$< \in \text{ if } \frac{2}{\in^2} < n - 1$$

$$< \in \text{ if } \frac{2}{\in^2} + 1 < n$$

$$< \in \text{ if } n > \frac{2}{\in^2} + 1$$

$$< \in \text{ if } n > m, \; m = \frac{2}{\in^2} + 1$$

$$|f_n| < \in \text{ if } n > m, \; m = \frac{2}{\in^2} + 1$$

$$|f_n - 0| < \in \text{ if } n > m, \; m = \frac{2}{\in^2} + 1$$

$\Rightarrow \qquad f_n \to 0$

$\Rightarrow \qquad (n)^{\frac{1}{n}} \to 0$

$\Rightarrow \qquad \sqrt[n]{n} \to 1$

2.4.4.Theorem : Show that every sequence converges to unique limit.

or

Prove that every convergent sequence converges to one and only one point.

Solution : Suppose $\{f_n\}$ converges to l and l', we show $l = l'$

Assume $l \neq l' \quad \Rightarrow \quad l - l' \neq 0 \quad \Rightarrow \quad |l - l'| > 0$

Let $\in = |l - l'|$. Clearly $\in > 0$

As $f_n \to l$, so for $\in > 0, \exists m_1 \in N$ such that $|f_n - l| < \in/2, \; \forall n > m_1$ ...(1)

Also, $f_n \to l'$, so for $\in > 0, \exists m_2 \in N$ such that $|f_n - l'| < \frac{\in}{2}, \; \forall n > m_2$ ...(2)

Choose $k = \min(m_1, m_2)$

Consider $|l-l'|=|(l-f_n)+(f_n-l)|$

$$\leq |l-f_n|+|f_n-l'|$$

$$<\frac{\in}{2}+\frac{\in}{2}, \forall n \geq k \quad \text{and using (1) and (2)}$$

$$=\in$$

$|l-l'|<|l-l'|$ using value of $\in$ which is not possible

Supposition is wrong.

Hence $l=l'$ $\Rightarrow$ $\{f_n\}$ converges to unique limit.

2.4.5.Exercise : Prove that every convergent sequence bounded but converse need not to be true.

Solution: Suppose $\{f_n\}$ is a convergent sequence. Let $f_n \rightarrow 1$

This means, for $\in > 0 \; \exists m \in \mathrm{N}$, such that $|f_n - l| < \in, \forall n \geq m$

$$l-\in < f_n < l+\in, \; \forall n \geq m \qquad ...(1)$$

Let $k=\min\{f_1, f_2, ...f_{m-1}, l-\in\}$ and $k'=\max\{f_1, f_2, f_3, f_4......f_{m-1}, l+\in\}$

Clearly, using this and (1) we see that

$$k < f_n < k', \; \forall n \in \mathrm{N}$$

$\Rightarrow$ $\{f_n\}$ is bounded.

(Definition of bounded sequence {see below})

Conversely, suppose $\{f_n\} = \{(-1)^{n-1}\} = \{-1, 1, -1, 1, ...\}$

Clearly, $\{f_n\}$ is bounded $-1 \leq f_n \leq 1, \forall n \in \mathrm{N}$

But $\lim_{n \to \infty} f_n$ is either 1 or –1 which is not possible.

As sequence always converges to unique limit

$\therefore$ $\{f_n\}$ is not convergent.

2.4.6. Bounded above Sequence A $\{f_n\}$ is said to be bounded above if there exist a real number $M \in R$ such that set $f_n \leq M, \forall n \in N$.

2.4.7. Bounded below Sequence A $\{f_n\}$ is said to be bounded below, if there exist a real number $m \in R$ such that $m \leq f_n, \forall n \in N$.

2.4.8.Bounded Sequence A $\{f_n\}$ is said to be bounded below, if there exist a real number m, $M \in R$ such that $m \leq f_n \leq M, \forall n \in N$.

Example : $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\right\}$ is bounded above and bounded below as, if $m = -1$ and $M = 1$. Then $-1 < f_n \leq 1, \forall n \in N$.

Example : { 1,–1,1,–1,1,–1,.......} is a bounded sequence as $-1 \leq f_n \leq 1, \forall n$.

Example : {1, 2, 3,.....} is bounded below as $-1 \leq f_n, \forall n \in N$.

But it is not bounded above as there doesn't exist any $m \in R$ such that $m \leq f_n, \forall n \in N$.

Example : {............–4, –3, –2, –1} is bounded above.

Here, $f_n \leq -1, \forall n \in N$ but there doesn't exist $m \in R$, such that $m \leq f_n, \forall n \in N$.

2.4.9. Exercise : Suppose $f_n \to l, g_n \to m,$ then show

(i) $f_n + g_n \to l + m$

(ii) $f_n - g_n \to l - m$

(iii) $\frac{f_n}{g_n} \to \frac{l}{m}$

(iv) $f_n g_n \to lm$

or

if $\lim_{n \to \infty} f_n = l, \lim_{n \to \infty} g_n = m,$ then

(i) $\lim_{n \to \infty} f_n + g_n = l + m$

(ii) $\lim_{n \to \infty} (f_n - g_n) = l - m$

(iii) $\lim_{n\to\infty} \frac{f_n}{g_n} = \frac{1}{m}, m \neq 0$ (iv) $\lim_{n\to\infty} f_n g_n = lm$

Solution : (i) To prove $f_n + g_n \to l + m$. Let $\in > 0$

Consider $|(f_n + g_n) - (l+m)| = |f_n - l| + |g_n - m|$ ...(1)

As $f_n \to 1$, so for $\in > 0, \exists m_1, \in \text{N}$ such that

$$|f_n - l| < \frac{\in}{2}, \forall n \geq m_1 \quad ...(2)$$

Also, $g_n \to m$, so for $\in > 0$, $\exists m_2 \in \text{N}$ such that

$$|g_n - m| < \frac{\in}{2}, \forall n \geq m_2 \quad ...(3)$$

Choose K = min (m_1, m_2)

Use (2), (3) in (1)

$$|(f_n + g_n) - (l+m)| < \frac{\in}{2} + \frac{\in}{2}, \quad \forall n \geq \text{K}$$

$$|(f_n + g_n) - (l+m)| < \in, \quad \forall n \geq \text{K}$$

$\Rightarrow \quad f_n + g_n \to l + m$

(ii) Let $\in > 0$

Consider $|(f_n - g_n) - (l-m)| = |(f_n - l) - (g_n - m)|$

$$= |f_n - l| + |(-1)(g_n - m)|$$

$$\leq |f_n - l| + |(-1)(g_n - m)| \quad ..(1)$$

As $f_n \to l$ so for $\in > 0, \exists m_1 \in \text{N}$ such that

$$| f_n - l | < \frac{\in}{2}, \ \forall n \geq m_1 \qquad ...(2)$$

Also $g_n \to m$ so for $\in > 0, \exists m_2 \in \mathrm{N}$ such that

$$| g_n - m | < \frac{\in}{2}, \ \forall n \geq m_2 \qquad ...(3)$$

Choose $k = \min(m_1, m_2)$

Use (2), (3) in (1)

$$|(f_n - g_n) - (l - m)| < \frac{\in}{2} + \frac{\in}{2}, \ \forall n \geq k$$

$$|(f_n - g_n) - (l - m)| < \in, \quad \forall n \geq k$$

$\Rightarrow \qquad f_n - g_n \to l - m$

(iii) Let $\in > 0$

Consider $\left| \frac{f_n}{g_n} - \frac{l}{m} \right| = \left| \frac{f_n m - l g_n}{g_n m} \right| = \left| \frac{f_n m - lm + lm - l g_n}{g_n m} \right|$

$$= \left| \frac{m(f_n - l) + (-l)(g_n - m)}{g_n m} \right|$$

$$= \left| \frac{m(f_n - l)}{g_n m} + \frac{(-l)(g_n - m)}{g_n m} \right|$$

$$\leq \left| \frac{m(f_n - l)}{g_n m} + \frac{(-l)(g_n - m)}{g_n m} \right|$$

$$\leq \left|\frac{(f_n - l)}{g_m}\right| + \left|\frac{-l(g_n - m)}{g_n m}\right|$$

$$= \frac{|(f_n - l)|}{|g_n|} + \frac{|l||(g_n - m)|}{|g_n||m|} \qquad ...(1)$$

As $\{g_n\}$ is convergent, so it is bounded, means $\exists k, k' \in \mathrm{R}$

$k \leq g_n \leq k', \;\; \forall n \in \mathrm{N}$

$$\Rightarrow \qquad \frac{1}{k} \geq \frac{1}{g_n} \geq \frac{1}{k'}$$

$$\Rightarrow \qquad \frac{1}{k'} \leq \frac{1}{g_n} \leq \frac{1}{k}$$

$$\Rightarrow \qquad \frac{1}{g_n} \leq \frac{1}{k} \;\Rightarrow\; \frac{1}{|g_n|} \leq \frac{1}{|k|} \text{ use in (1)}$$

$$\left|\frac{f_n}{g_n} - \frac{1}{m}\right| \leq \frac{|f_n - l|}{|k|} + \frac{|l||g_n - m|}{|k||m|} \qquad ...(2)$$

As $f_n \to l$ so for $\in > 0, \; \exists m_1 \in \mathrm{N}$ such that $|f_n - l| < \frac{\in|k|}{2}, \;\; \forall n \in m_1$ ...(3)

Also $g_n \to m$, so for $\in > 0, \exists m_2 \in \mathrm{N}$ such that $|g_n - m| < \frac{\in|k||m|}{2|l|}, \;\; \forall n \in m_2$

...(4)

Let $m_0 = \min(m_1, m_2)$

Use (3), (4) in (2)

$$\left|\frac{f_n}{g_n} - \frac{l}{m}\right| < \frac{1}{|k|}\frac{\in|k|}{2} + \frac{|l|}{|m||k|}\frac{\in|k||m|}{2|l|} \;\; \forall n \geq m_0$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

$\Rightarrow \qquad \frac{f_n}{g_n} \to \frac{1}{m}$

(iv) Consider $|f_n g_n - lm| = |f_n g_n - f_n m + f_n m - lm|$

$$= |f_n (g_n - m) + m(f_n - l)|$$

$$\leq |f_n (g_n - m)| + |m(f_n - l)|$$

$$= |f_n||g_n - m| + |m||f_n - l| \quad ...(1)$$

As $\{f_n\}$ is convergent.

Sequence, so it is bounded, so $\exists k, k' \in \mathrm{R}$ such that $k \leq f_n \leq k', \forall n \in \mathrm{N}$

i.e. $f_n \leq k' \Rightarrow |f_n| \leq |k'|$

Use in (1)

$$|f_n g_n - lm| \leq |k'||g_n - m| + |m||f_n - l| \quad ...(2)$$

As $f_n \to l$ so for $\in > 0$, $\exists m_1 \in \mathrm{N}$ such that

$$|f_n - l| < \frac{\in}{2|m|}, \quad \forall n \geq m_1 \quad ...(3)$$

Also $g_n \to m$ so for $\in > 0, \exists m_2 \in \mathrm{N}$ such that

$$|g_n - m| \leq \frac{\in}{2|k'|} \quad \forall n \geq m_2 \quad ...(4)$$

Choose $m_0 = \min(m_1, m_2)$

Using (3), (4) in (2), we get

$$|f_n g_n - lm| < |k'|\frac{\in}{2|k'|} + |m|\frac{\in}{2|m|}, \ \forall n \geq m_0$$

$$< \frac{\in}{2} + \frac{\in}{2} = \in$$

$$\Rightarrow \qquad |f_n g_n - lm| < \in, \ \forall n \geq m_0$$

$$\Rightarrow \qquad f_n g_n \to lm$$

2.4.10. Note : The converse of the above expression need not to be true *i.e.*

If $\{f_n\}, \{g_n\}$ be sequence such that their

(1) sum (2) difference (3) product

(4) quotient are convergent but sequences $\{f_n\}, \{g_n\}$ need not to be convergent.

Solution : (1) Consider $\{f_n\} = \{1, -1, 1, -1, ...\}$ and $\{g_n\} = \{-1, 1, -1, 1, ...\}$

Clearly their sum $\{f_n + g_n\} = \{0, 0, 0, 0.........\} \to 0$

But neither $\{f_n\}$ nor $\{g_n\}$ is not convergent.

(2) Let $\{f_n\} = \{1, -1, 1-1, 1, -1....\}$ and $\{g_n\} = \{1, -1, 1-1, 1, -1....\}$

Then $\{f_n - g_n\} = \{0, 0, 0, 0.........\} \to 0$

But neither $\{f_n\}$ nor $\{g_n\}$ is convergent.

(2) Let $\{f_n\} = \{1, -1, 1, -1, 1, -1, ...\}$

and $\{g_n\} = \{1, -1, 1, -1, 1, -1, ...\}$

and $\{f_n - g_n\} = \{0, 0, 0, 0...........\} \to 0$

But neither $\{f_n\}$ nor $\{g_n\}$ is convergent

(3) Take $\{f_n\} = \{1, -1, 1, -1, ...\}$

and $\{g_n\} = \{-1, 1, -1, 1, ...\}$

$\Rightarrow \quad \{f_n g_n\} = \{-1, -1, -1, -1...........\} \to -1$, as $n \to \infty$

But neither $\{f_n\}$ nor $\{g_n\}$ is convergent

(4) Take $\{f_n\} = \{-1, 1, -1, 1, ...\}$

And $\{g_n\} = \{-1, 1, -1, 1, ...\}$

$$\Rightarrow \quad \left\{\frac{f_n}{g_n}\right\} = \{1, 1, 1, 1, \ldots\ldots\ldots\ldots\} \to 1, \text{ as } n \to \infty$$

But neither $\{f_n\}$ nor $\{g_n\}$ is convergent.

2.5. EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

Q.1. Prove that every convergent sequence bounded but converse need not to be true.

Q.2. Define convergent sequence prove that is convergent sequence & converging to 1.

Q.3. Suppose $f_n \to l, g_n \to m$, then show

(i) $f_n + g_n \to l + m$ (ii) $f_n - g_n \to l - m$

(iii) $\frac{f_n}{g_n} \to \frac{l}{m}$ (iv) $f_n g_n \to lm$

Q.4. If $\{f_n\}$, $\{g_n\}$ be sequence such that their

(1) sum (2) difference (3) product

(4) quotient are convergent but sequence $\{f_n\}$, $\{g_n\}$ need not to be convergent.

2.6. SUGGESTED READING

The students are advised to go through following references for details

2.7. REFERENCES

(1) Real analysis by by J.N. Kapur & H.C. Saxena, S.Chand & Co.

(2) Real analysis by J.N. Sharma & A.R.Vashishtha,Krishna Publication, New Delhi.

(3) Real analysis by Richard R. Goldberg, Oxford & IBH Publication Co. Pvt. Ltd. New Delhi.

(4) A text Book of Real Analysis by Sunil Gupta, Narinder Kumar, Malhotra Brothers Pacca Danga, Jammu.

(5) Real analysis by Jagdish Parsad Mittal & Neeraj Doda, Sharma Publication, Jallandhar.

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Q.1. Prove that every convergent sequence bounded but converse need not to be true.

Q.2. Define convergent sequence prove that is convergent sequence & converging to 1.

Q.3. Suppose $f_n \to l, g_n \to m$, then show

(i) $f_n + g_n \to l + m$ (ii) $f_n - g_n \to l - m$

(iii) $\frac{f_n}{g_n} \to \frac{l}{m}$ (iv) $f_n g_n \to lm$

Q.4. If $\{f_n\}$, $\{g_n\}$ be sequence such that their

(1) sum (2) difference (3) product

(4) quotient are convergent but sequence $\{f_n\}$, $\{g_n\}$ need not to be convergent.

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MONOTONE SEQUENCE

3.1. Introduction : In this lesson the continuation of convergence of sequence of functions are discussed.

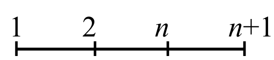
3.2 Objectives : Objective of studying this lesson is to give idea of which sequence is increasing & decreasing. Also the concept of famous Nested Interval Property/ Cantor intersection theorem are reported in this lesson.

3.3. MONOTONE SEQUENCE

3.3.1. Monotone Increasing : A $\{f_n\}$ is said to be monotone increasing

If $n \geq m$, $f_m \leq f_n$

3.3.2. Decreasing sequence : A $\{f_n\}$ is said to be decreasing if $f_n > f_{n+1}$, $\forall n \in \mathbb{N}$



A $\{f_n\}$ is decreasing, if $n \geq m$, then $f_m > f_n$.

3.3.3. Monotone decreasing : A $\{f_n\}$ is said to be monotone decreasing if $n \geq m$ then $f_m \geq f_n$.

3.3.4. Monotone : A sequence which is either Monotone increasing or Monotone decreasing is called a monotone sequence.

3.3.5. Examples

1. $\{f_n\} = \{n\} = \{1, 2, 3, 4, 5, ...\}$ is an increasing sequence

   as $f_1 < f_2 < f_3 < f_4 .... < f_n < f_{n+1} < ....$

2. $\{f_n\} = \{1, 2, 3, 3, 4, 5, 6, 6, 7.............\}$ is monotonic increasing

as $f_1 < f_2 < f_3 = f_4 < f_5 < f_6 = f_7 < f_8$

3. $\{f_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\right\}$ is decreasing,

Since $f_1 > f_2 > f_3 > f_4 >$..............................

4. $\{f_n\} = \{(-1)^{n-1}\} = \{1, -1, 1, -1, ...\}$ is neither increasing nor decreasing because

$f_1 > f_2 < f_3 > f_4 < f_5 >$................

3.3.6. State And Prove Monotone Convergence Theorem. Every Monotone increasing bounded above sequence is always convergent

Proof: Suppose $\{f_n\}$ is monotone increasing & bounded above sequence. We show $\{f_n\}$ is convergent.

As $\{f_n\}$ is Monotone increasing, so for $n \geq m$, we have $f_m \leq f_n$ ...(1)

also $\{f_n\}$ is bounded above, so, let l is l.u.b of $\{f_n\} \Rightarrow f_n \leq l$

Let $\in > 0$

Then $f_n \leq l < l + \in$ ...(2)

As $1 - \in < l$ so there exist so many entries between $l - \in$ and l.

Let one of these entries be f_m *i.e.* $1 - \in < f_m < 1$

$$1 - \in < f_m < 1$$

$$1 - \in < f_m$$

Combine with (1), we get

$$1 - \in < f_m \leq f_n$$

or $$1 - \in < f_n \quad ...(3)$$

Combine (2) and (3), we have

$$1 - \in < f_n < l + \in, n \geq m$$

$$\Rightarrow \quad |f_n - l| < \in, \ \forall n \to m$$

$$\Rightarrow \quad f_n \to l$$

3.3.7. Corollary : Every monotone decreasing and bounded below sequence bounded is convergent

Proof : Let $\{f_n\}$ be m.d and bounded below, then $\{-f_n\}$ becomes monotone increasing + bounded above.

Hence, by above theorem $\{-f_n\}$ is convergent

$-\{-f_n\}$ is convergent implies $\{f_n\}$ is convergent.

3.4. CAUCHY SEQUENCE

3.4.1. Definition : Cauchy sequence : A sequence $\{f_n\}$ is said to be a Cauchy if $\in > 0$, $\exists\, m \in N$ such that $|f_n - f_m| < \in,\ \forall\, n \geq m$

or

A $\{f_n\}$ is said to be Cauchy if $\in > 0$, $\exists\ p \in N$ such that $|f_n - f_m| < \in,\ \forall n, m \geq p$.

Notation : If $\{f_n\}$ is Cauchy, then $\lim\limits_{\substack{n\to\infty \\ m\to\infty}} |f_n - f_m| = 0$.

3.4.2. Example $\left\{\frac{1}{n}\right\}$ is Cauchy.

Solution : $f_n = \frac{1}{n} \quad \Rightarrow \quad f_m = \frac{1}{m}$

$$\lim_{\substack{n\to\infty \\ m\to\infty}} |f_n - f_m| = \lim_{n,m\to\infty} \left|\frac{1}{n} - \frac{1}{m}\right| \to 0$$

Example : Take $\{f_n\} = \{n^2\}$, then it is not Cauchy.

Since $\lim\limits_{n,m\to\infty} |f_n - f_m| = |n^2 - m^2| = 0 \quad \not\to 0$

3.4.3. Example : $\{f_n\} = \{-1, 1, -1, 1, \ldots\ldots\ldots\ldots\}$, then

$|f_1 - f_2| = |1 - (-1)| = 2 \quad \not\to 0$

3.4.4. Theorem : Every convergent sequence is Cauchy.

Proof : Suppose $\{f_n\}$ is a convergent sequence.

Let it converges to l *i.e.* $f_n \to l$

This means, $\in > 0, \exists m \in \mathrm{N}$ such that

$$| f_n - l | < \frac{\in}{2}, \ \forall n \geq m \qquad ...(1)$$

As (1) is true for all $n \geq m$

In particular, for $n = m$ *i.e.* $| f_m - l | < \frac{\in}{2}$ ...(2)

Consider $| f_n - f_m | = | f_n - l + l - f_m |$

$$\leq | f_n - l | + | l - f_n |$$

$$< \frac{\in}{2} + \frac{\in}{2}, \quad \forall n \geq m \qquad \text{(Using (1) and (2))}$$

$$< \in, \quad \forall n \geq m$$

$\Rightarrow$ $\{f_n\}$ is Cauchy

3.4.5. Exercise : Prove that every Cauchy sequence is bounded.

Solution : Let $\{f_n\}$ be a Cauchy sequence. Then for $\in > 0, \exists m \in \mathrm{N}$ such that

$$| f_n - f_m | < \in, \quad \forall n \geq m$$

$$f_m - \in < f_n < f_m + \in, \quad \forall n \geq m \qquad ...(1)$$

Let $k = \min\{f_1, f_2, ... f_{m-1}, f_m - \in\}$

$k' = \max\{f_1, f_2, ... f_{m-1}, f_m + \in\}$

Using (1) we see $k < f_n < k', \quad \forall n \in \mathrm{N}$

$\Rightarrow$ $\{f_n\}$ is bounded.

3.6. BALZANO WEIRSTRASS THEOREM

3.6.1. Statement : Every infinite bounded sequence has a convergent subsequent.

3.6.2. Theorem : Every Cauchy sequence is convergent.

Proof : Let $\{f_n\}$ be a Cauchy sequence. For $\in > 0$ $\exists$ $m \in \mathrm{N}$ such that

$$| f_n - f_m | < \frac{\in}{3}, \quad \forall\, n \geq m \qquad ...(1)$$

As $\{f_n\}$ is Cauchy sequence, so it is bounded.

Let $S = \{ f_n : n \in \mathrm{N} \}$. Then S is an infinite bounded set.

Then by B.W theorem, $\{f_n\}$ has a convergent subsequence say $\{f_{n_k}\}$

As $\{ f_{n_k} \}$ is convergent subsequence. So, let it converges to l *i.e.* $f_{n_k} \to l$

Then for $\in > 0$, $\exists\, p \in \mathrm{N}$ such that $| f_{n_k} - l | < \frac{\in}{3}, \ \forall\, n_k \geq p \qquad ...(2)$

Let If $m > n_k$ from (1), $| f_m - f_{n_k} | < \frac{\in}{3}, \ n \geq n_k$

Consider $| f_n - l |$

$$\leq | f_n - f_m | + | f_m - f_{n_k} | + | f_{n_k} - l |$$

$$< \frac{\in}{3} + \frac{\in}{3} + \frac{\in}{3} < \in, \quad \forall\ n \geq m$$

$\Rightarrow \quad f_n \to l$

$\{f_n\}$ is convergent.

3.6.3. Remark : Every bounded sequence need not to be Cauchy

Proof : Take $\{f_n\} = \{(-1)^{n-1}\} = \{1, -1, 1, -1, ...\}$

Clearly, $\{f_n\}$ is bounded sequence as $-1 \leq f_n \leq 1, \forall\, n \geq \mathrm{N}$

But $\{f_n\}$ is not Cauchy as $| f_1 - f_2 | = | 1 - (-1) | = 2 \not\to 0$

3.7. NESTED INTERVAL PROPERTY OR CANTOR INTERSECTION THEOREM

3.7.1. Nested Sequence : A sequence $\{I_n\}$ where $I_n = [a_n, b_n], \ \forall\, n \in \mathrm{N}$ of closed intervals is said to be a nested sequence if either $I_n \subset I_{n+1}$ or $I_n \supset I_{n+1}, \ \forall\, n \in \mathrm{N}$.

3.7.2. Statement : Let $\{I_n\}$ where $I_n = [a_n, b_n]$, $\forall n \in N$ be such that

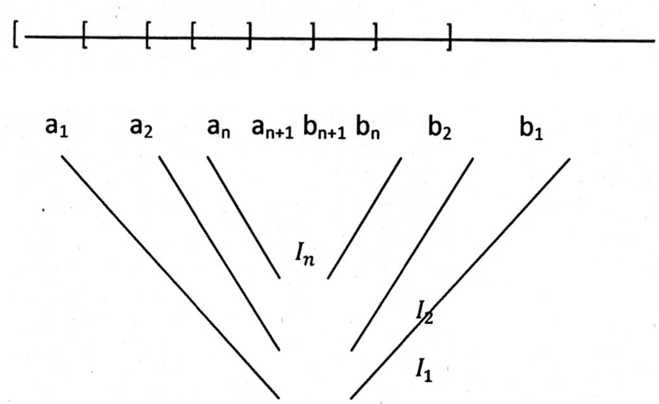
(i) $\{I_n\}$ is nested

(ii) $\lim_{n\to\infty} |I_n| = 0$, then $\bigcap_{n=1}^{\infty} I_n \neq \varphi$

Proof : (i) Let $\{I_n\}$, where $I_n = [a_n, b_n]$, $\forall n \in N$ be nested

This means, $I_n \supset I_{n+1}$, $\forall n \in N$

i.e. $I_1 \supset I_2 \supset I_3 \ldots\ldots\ldots\ldots \supset I_n \supset I_{n+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



$\Rightarrow \quad [a_1, b_1] \supset [a_2, b_2] \supset \ldots\ldots\ldots\ldots\ldots\ldots\ldots \supset [a_n, b_n] \supset [a_{n+1}, b_{n+1}] \supset \ldots\ldots\ldots$

From diagram, it is clear that $a_1 < a_2 < a_3 \ldots\ldots\ldots\ldots < a_n < a_{n+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Now, from the diagram, we see

$$a_1 < a_2 < a_3 < \ldots\ldots\ldots\ldots < a_n < a_{n+1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots < b_1$$

i.e. $a_n < b_1$, $\forall n \in N \Rightarrow \{a_n\}$ is bounded above.

Since $\{a_n\}$ is increasing and bounded above, it follows by monotone convergent theorem $\{a_n\}$ is converges to l

i.e. $a_n \to l$ or $\lim_{n\to\infty} a_n = l$ ...(i)

Again, from diagram, we see

$$b_1 > b_2 > b_3 > \ldots\ldots\ldots > b_n \ldots\ldots\ldots$$

$\Rightarrow \quad \{b_n\}$is decreasing

Also $b_1 > b_2 > b_3 > \ldots\ldots\ldots \geq a_1$

$\Rightarrow \quad b_n > a_1$

$\Rightarrow \quad \{b_n\}$ is bounded below

As $\{b_n\}$ is decreasing and bounded below, it follows that $\{b_n\}$ is convergent

Let it converges to m *i.e.* $b_n \to m$ or $\lim_{n\to\infty} b_n = m$ ...(2)

It is given $\lim_{n\to\infty} |l_n| = 0$ *i.e.* $\lim_{n\to\infty} |[a_n, b_n]| = 0$

$$\lim_{n\to\infty} (b_n - a_n) = 0$$

$$\lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = 0 \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

$(l = m)$

$\{a_n\}$ and $\{b_n\}$ converges to same limit

$\Rightarrow \quad a_n \to l \leftarrow b_n$

$\Rightarrow \quad a_n \leq l \leq b_n, \quad \forall\, n \in \mathrm{N}$ [because convergent sequence are bounded]

$\Rightarrow \quad l \in [a_n, b_n], \quad \forall\, n \in \mathrm{N}$

$\Rightarrow \quad \bigcap_{n=1}^{\infty} \mathrm{I}_n \neq \varphi$

3.7.3. Example Show that $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is convergent and converging to e, $2 < e < 3$.

or

Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e,\ 2 < e < 3$

Proof let $f_n = \left(1+\frac{1}{n}\right)^n$

To show $\{f_n\}$ converges to e, we show

(i) $\{f_n\}$ is monotone increasing

(ii) $\{f_n\}$ is bounded above.

Here $f_n = \left(1+\frac{1}{n}\right)^n$

$$= (1)^n + n_{c_1}(1)^{n-1}\left(\frac{1}{n}\right) + n_{c_2}(1)^{n-2}\left(\frac{1}{n}\right)^2 + ... + n_{c_n}(1)^n\left(\frac{1}{n}\right)$$

$$= 1 + \left(\frac{n}{n}\right) + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + ... + \frac{n(n-1)(n-1)........[n-(n-1)]}{n!}\frac{1}{n^n}$$

$$= 1 + \left(\frac{n}{n}\right) + \frac{1}{2!}\frac{n(n-1)}{n.n} + \frac{1}{3!}\frac{n(n-1)(n-2)}{n.n} + ... + \frac{1}{n!}\left[\frac{n(n-1)(n-2).....n-(n-1)}{n.n.n......n}\right]$$

$$=1+1+\frac{1}{2!}\left(\frac{n}{n}\right)\left(\frac{n-2}{n}\right)+\frac{1}{3!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)+....+\frac{1}{n!}\left[\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)....\left(\frac{n-(n-1)}{n}\right)\right]$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + ... + \left(1-\frac{n-1}{n}\right) \qquad ...(1)$$

$\Rightarrow \quad f_n > 2$ or $2 < f_n$ ...(2)

Consider

$$f_{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right)+\frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)+......+\frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right).....$$

$$\left(1-\frac{2}{n+1}\right) + ... + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)........\left(1-\frac{n}{n+1}\right) \qquad ...(3)$$

On comparing each term of f_n with each term of f_{n+1}, we see

$$n < n+1 \quad \Rightarrow \quad \frac{1}{n} > \frac{1}{n+1} \quad \Rightarrow \quad \frac{1}{n} < -\frac{1}{n+1}$$

$$\Rightarrow \qquad 1-\frac{1}{n} < 1-\frac{1}{n+1}$$

Similarly, $1-\frac{2}{n} < 1-\frac{2}{n+1}$, and so on $1-\frac{n-1}{n} < 1-\frac{n}{n+1}$

We see $f_n \leq f_{n+1}, \quad \forall\, n \in \mathrm{N}$

$\Rightarrow \qquad \{f_n\}$ is monotone increasing

Also, we know $1-\frac{1}{n} < 1,\ 1-\frac{2}{n} < 1, \ldots\ldots\ldots\ldots\ldots\ 1-\frac{n-1}{n} < 1$

Use this in (2)

$$f_n \leq 1+1+\frac{1}{2!}(1)+\frac{1}{3!}(1)(1)+\ldots\ldots+\frac{1}{n!}(1)(1).(1)$$

$$= 1+\left(1+\frac{1}{2}+\frac{1}{6}+\ldots\ldots+\frac{1}{n!}\right) \qquad \ldots(4)$$

Again $6 > 4 \;\Rightarrow\; \frac{1}{2} < \frac{1}{4}$ or $\frac{1}{6} < \frac{1}{2^2}$

And so on $\frac{1}{n!} < \frac{1}{2^{n-1}}$

Use in (4)

$$f_n \leq 1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots+\frac{1}{2^{n-1}}\right)$$

$$= 1+\frac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}=\frac{1}{2}}$$

$\Rightarrow \qquad f_n \le 1 + 2\left(1 - \frac{1}{2^n}\right)$

$$= 3 - \frac{1}{2^{n-1}} \le 3 \quad \Rightarrow \quad \{f_n\} \text{ is bounded ab ove.}$$

Since $\{f_n\}$ is monotone increasing and bounded above,so by monotone increasing the sequence $\{f_n\}$ is convergent.

Take $\lim_{n \to \infty}$ to both side $\lim_{n \to \infty} f_n \le \lim_{n \to \infty}\left(3 - \frac{1}{2^{n-1}}\right) = 3$

or $\qquad \lim_{n \to \infty} f_n \le 3 \qquad$ ...(5)

Consider (2), (5) we see

$$2 < \lim_{n \to \infty} f_n < 3$$

or $\qquad \lim_{n \to \infty} f_n = e$, where $2 < e < 3$

or $\qquad \lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e,\ 2 < e < 3$

$\Rightarrow \qquad \left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is convergent, converging to e, $2 < e < 3$.

3.8. EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

Q.1. Prove that every Cauchy sequence converges iff it is convergent.

Q.2. State & prove Monotone convergence Theorem.

3.9. SUGGESTED READING

The students are advised to go through following references for details

3.10. REFERENCES

1. Real analysis by by J.N. Kapur & H.C. Saxena, S.Chand & Co.
2. Real analysis by J.N. Sharma & A.R.Vashishtha,Krishna Publication, New Delhi.
3. Real analysis by Richard R. Goldberg, Oxford & IBH Publication Co. Pvt. Ltd. New Delhi.
4. A text Book of Real Analysis by Sunil Gupta, Narinder Kumar, Malhotra Brothers Pacca Danga, Jammu.
5. Real analysis by Jagdish Parsad Mittal & Neeraj Doda, Sharma Publication, Jallandhar.

3.11. MODEL TEST PAPER

Q.1. Define a Cauchy sequence & Show that sequence $1/n$ is a Cauchy sequence.

Q.2. State & prove Monotone convergence Theorem.

Q.3. Prove that $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is convergent and converging to e, $2 < e < 3$.

Q.4. Prove that every Cauchy sequence is convergent.

Q. 5. Prove that every convergent sequence is Cauchy.

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INFINITE SERIES

4.1. Introduction : In this lesson the concept of infinite series of functions are discussed.

4.2 Objectives : Objective of studying this lesson is to familiar students about the concept of positive infinite series..

4.3. INFINITE SERIES

A series of the type $f_1 + f_2 + f_3 + ...$ is called an infinite series. It is denoted by $\sum_{n=1}^{\infty} f_n$ or $\sum f_n$.

If all terms of series are positive, then it is called a series of positive term.

4.3.1. Sequence of partial sum : Let $\sum f_n = f_1 + f_2 + f_3 + ...$ be infinite series

Consider $S_1 = f_1$

$S_2 = f_1 + f_2$

$S_3 = f_1 + f_2 + f_3$

:

:

:

$S_n = f_1 + f_2 + f_3 .... + f_n$

:

:

:

then $\{S_1, S_2, S_3, \ldots\ldots\ldots S_n, \ldots\ldots\}$ is called sequence of partial sum.

4.4. CONVERGENT SERIES

Let $\sum f_n$ be an infinite series with $\{S_n\}$ be sequence of partial sum where

$S_n = f_1 + f_2 + \ldots\ldots + f_n$

We say, $\sum f_n$ converges to l, if $\{S_n\}$ of partial sum converges to l

i.e. if $\lim_{n\to\infty} S_n = l$, then $\sum f_n \to l$.

4.4.1. Example : Show that series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is convergent

Solution : $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}=\frac{1}{2}} \qquad \left[\because \frac{a(1-r^n)}{1-r}, r<1\right]$$

$$= \lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right)$$

Here, $S_n \to 2$

So, series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ also converges to 2.

4.4.2. Example : Show that series $\sum_{n=1}^{\infty} n$ is divergent.

Solution : Here, $S_n = \sum_{n=1}^{\infty} n = 1 + 2 + 3 + ... + n$

$S_1 = 1,\ S_2 = 1 + 2,\ S_3 = 1 + 2 + 3$

$S_n = 1 + 2 + 3 + ... + n$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 + 2 + 3 + ...+ n)$$

$$= \lim_{n\to\infty} \frac{n(n+1)}{2} \to \infty$$

$\therefore$ $\{S_n\}$ diverges, so $\sum_{n=1}^{\infty} n$ also diverges.

4.4.3. Example: An infinite series $\sum_{n=1}^{\infty} u_n$ is convergent, then $u_n \to 0$ as $n \to \infty$.

But converse need not to be true.

4.5. COMPARISON TEST

4.5.1. 1st comparison test

Let $\sum u_n$ be an infinite series

Choose, $\sum v_n$ such that

(i) $\lim_{n\to\infty} \frac{u_n}{v_n} =$ a finite number

(ii) $u_n \le v_n,\ \forall\, n \in \mathrm{N}$. Then, $\sum u_n$ is convergent. If $\sum v_n$ is converget.

4.5.2. 2nd comparison test

Let $\sum u_n$ be an infinite series.

Choose, $\sum v_n$ such that

(i) $\lim_{n\to\infty} \frac{u_n}{v_n} =$ a finite number

(ii) $u_n \geq v_n, \ \forall n \in N$

Then, $\sum u_n$ is divergent if $\sum v_n$ is divergent.

4.6. EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

Q.1. Show that series $\sum_{n=1}^{\infty} 2n$ is divergent.

Q.2. State 1st Comparison test.

Q.3. state 2nd Comparison test.

4.7. SUGGESTED READING

The students are advised to go through following references for details

4.8. REFERENCES

(1) Real analysis by by J.N. Kapur & H.C. Saxena, S.Chand & Co.

(2) Real analysis by J.N. Sharma & A.R.Vashishtha,Krishna Publication, New Delhi.

(3) Real analysis by Richard R. Goldberg, Oxford & IBH Publication Co. Pvt. Ltd. New Delhi.

(4) A text Book of Real Analysis by Sunil Gupta, Narinder Kumar, Malhotra Brothers Pacca Danga, Jammu.

(5) Real analysis by Jagdish Parsad Mittal & Neeraj Doda, Sharma Publication, Jallandhar.

4.9. MODEL TEST PAPER

Q.1. Define an infinite series & show that how it converges or diverges.

Q.2. State 1st Comparison test.

Q.3. State 2nd Comparison test.

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p - SERIES TEST

5.1. Introduction : In this lesson the idea of how a series converges or diverges is discussed.

5.2 Objectives : Objective of studying this lesson is to explain the concept of convergence of an infinite series.

5.3. *p*-SERIES TEST

5.3.1. State and prove p–Series Test

Statement : $\sum_{n=1}^{\infty} \frac{1}{n^p}$ be an infinite series, then it converges if $p > 1$ and diverges if $p \le 1$.

Proof : Case I : When $p > 1$

Given series

$$\sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + ...$$

$$= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + ... \qquad ...(1)$$

As $3 > 2 \Rightarrow 3^p > 3^p \Rightarrow \frac{1}{3^p} \le \frac{1}{2^p}$

$5 > 4 \Rightarrow 5^p > 4^p \Rightarrow \frac{1}{5^p} \le \frac{1}{4^p}$

$$6 > 4 \quad \Rightarrow \quad 6^p > 4^p \quad \Rightarrow \quad \frac{1}{6^p} \le \frac{1}{4^p}$$

$$7 > 4 \quad \Rightarrow \quad 7^p > 4^p \quad \Rightarrow \quad \frac{1}{7^p} \le \frac{1}{4^p}$$

And so on use in (1)

$$\sum \frac{1}{n^p} \le 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \ldots\ldots\ldots$$

$$= 1 + \left(\frac{2}{2^p}\right) + \left(\frac{4}{4^p}\right) + \ldots\ldots\ldots$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \ldots\ldots\ldots$$

$$= 1 + \frac{1}{2^{p-1}} + \frac{1}{(2^{p-1})^2} + \ldots\ldots\ldots \qquad \ldots(2)$$

As series on R.H.S of (2) is a G.P series with C.R = $\frac{1}{2^{p-1}} < 1$

So it converges, thus by 1st comparison test, series on L.H.S also converges.

Case II : When $p = 1$

$$\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots\ldots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots\ldots\ldots\ldots \qquad \ldots(3)$$

$$3 < 4 \quad \Rightarrow \quad \frac{1}{3} > \frac{1}{4}$$

$$5 < 8 \quad \Rightarrow \quad \frac{1}{5} > \frac{1}{8}$$

$$6 < 8 \Rightarrow \frac{1}{6} > \frac{1}{8}$$

$$7 < 8 \Rightarrow \frac{1}{7} > \frac{1}{8} \quad \text{and so on}$$

Use in (3), we get

$$\sum \frac{1}{n} \geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \ldots\ldots\ldots$$

$$> \frac{1}{2} + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \ldots\ldots\ldots \qquad \left[\because 1 > \frac{1}{2}\right] \qquad \ldots(4)$$

Consider $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots.$

Here, $S_n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots\ldots. + \frac{1}{2} = \frac{n}{2}$

$$\therefore \quad \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n}{2} = \infty$$

Thus, series on R.H.S of (4) diverges, so by 2nd comparison test, series on L.H.S also diverges.

Case III : When $p < 1$. Then, clearly $n \geq n^p$

$$\frac{1}{n} \leq \frac{1}{n^p} \quad \text{or} \quad \frac{1}{n^p} \geq \frac{1}{n}$$

Take both side, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \geq \sum \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent because series on R.H.S is divergent by case II.

Example 5.3.2. Test the convergence of the following series :

(i) $1+\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\text{...........}$ (ii) $\sum \frac{n(n+1)}{(n+2)(n+3)(n+4)}$

Solution : (i) $1+\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\text{...........}$

$$= 1+\sum_{n=1}^{\infty} u_n$$

Where $u_n = \frac{1}{n(n+3)} = \frac{1}{n\,.\,n\left(1+\frac{3}{n}\right)} = \frac{1}{n^2\left(1+\frac{3}{n}\right)}$

Choose $v_n = \frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{1}{n^2\left(1+\frac{3}{n}\right)}}{\frac{1}{n^2}} = 1$$

Now $\sum v_n = \sum \frac{1}{n^2}$ is convergent by *p*-series test.

By comparison Test $\sum u_n$ also convergent.

Hence, 1 + $\sum u_n$ also convergent.

(ii) $\sum \frac{n(n+1)}{(n+2)(n+3)(n+4)} = \sum u_n$

$$u_n = \frac{n(n+1)}{(n+2)(n+3)(n+4)} = \frac{n\,.\,n\left(1+\frac{1}{n}\right)}{n^3\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)}$$

Choose $v_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\dfrac{1+\frac{1}{n}}{n\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)}}{\frac{1}{n}}$$

$$\to \frac{1}{(1)(1)(1)} = 1$$

Now, $\sum v_n = \sum \frac{1}{n}$ is divergent by *p*–series test.

So, by comparing test $\sum u_n$ also diverges.

5.4. D–ALMBERT'S RATIO TEST

Let $\sum u_n$ be an infinite series such that $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} - l$

Then, series

(i) $\sum u_n$ is convergent if $l > 1$

(ii) $\sum u_n$ is divergent. If $l < 1$

(iii) Test fails if $l = 1$

Proof : As $\frac{u_n}{u_{n+1}} \to l,$ so far $\in > 0$, there exists $m \in$N, such that

$$\left| \frac{u_n}{u_{n+1}} - l \right| < \in, \forall\, n \geq m$$

$$l - \in < \frac{u_n}{u_{n+1}} < l + \in, \quad \forall\, n \geq m \qquad ...(1)$$

Case I : When $l > 1$ then $1 < l - \in < 1$

From (1)

$$l - \in < \frac{u_n}{u_{n+1}}, \quad \forall\, n \geq m$$

Put $\quad n = m,\ m+1,\ m+2, ..., n-1$

We get

$$\left.\begin{array}{l} l - \in < \dfrac{u_m}{u_{m+1}} \\[2ex] l - \in < \dfrac{u_{m+1}}{u_{m+2}} \\[2ex] l - \in < \dfrac{u_{m+2}}{u_{m+3}} \\ : \\ : \\ l - \in < \dfrac{u_{n-1}}{u_n} \end{array}\right\}$$

Multiplying these $(n - m)$ inequalities, we get

$$(l - \in)(l - \in)......(l - \in) < \frac{u_m}{u_{m+1}} \frac{u_{m+1}}{u_{m+2}} \frac{u_{m+2}}{u_{m+3}} .... \frac{u_{n-1}}{u_n}$$

$$(l - \in)^{n-m} < \frac{u_m}{u_n}$$

$$u_n (l - \in)^{n-m} < u_m$$

$$u_n < \frac{u_m}{(1 - \in)^{n-m}}$$

$$u_n < \frac{u_m}{(1 - \in)^{-m} (1 - \in)^n}$$

$$\sum_{n=1}^{\infty} u_n < \sum_{n=1}^{\infty} \frac{u_m}{(1 - \in)^{-m} (1 - \in)^n} = \frac{u_m}{(1 - \in)^{-m}} \sum_{n=1}^{\infty} \frac{1}{(1 - \in)^n} \qquad ...(2)$$

Consider

$$\sum_{n=1}^{\infty}\frac{1}{(1-\in)^n}=\frac{1}{1-\in}+\frac{1}{(1-\in)^2}+\frac{1}{(1-\in)^3}+\text{.........}$$ is G.P series with c.r = $\frac{1}{1-\in}<1$

(as $1 < 1 - \in$ or $1 > \frac{1}{1-\in}$ or $\frac{1}{1-\in}<1$)

So this series $\sum\frac{1}{(l-\in)^n}$ is convergent.

Hence, series on L.H.S of (2)

$\sum u_n$ also convergent for $l > 1$

Case II : When $l < 1$. Clearly $l < l + \in < 1$

From (1)

$$\frac{u_n}{u_{n+1}}<l+\in,\ \forall\, n\geq m$$

Put $n = m, m + 1, m + 2,$...............$n - 1$, we get

$$\left.\begin{aligned}&\frac{u_n}{u_{n+1}}<l+\in\\&\frac{u_n}{u_{n+2}}<l+\in\\&\frac{u_n}{u_{n+3}}<l+\in\\&:\\&:\\&:\\&\frac{u_{n-1}}{u_n}<l+\in\end{aligned}\right\}\quad (n-m)\text{ inequalities.}$$

$$\frac{u_m}{u_{m+1}} \cdot \frac{u_{m+1}}{u_{m+2}} \cdot \frac{u_{m+2}}{u_{m+3}} \cdots\cdots \frac{u_{n-1}}{u_n} < (l+\in)(l+\in)(l+\in)\cdots\cdots(l+\in)$$

$$\frac{u_m}{u_n} < (l+\in)^{n-m}$$

$$u_m < u_n\,(l+\in)^{n-m}$$

$$\frac{u_m}{(l+\in)^{n-m}} < u_n$$

or
$$u_n > \frac{u_m}{(l+\in)^n.(l+\in)^{-m}}$$

$$\sum_{n=1}^{\infty} u_n > \sum_{n=1}^{\infty} \frac{u_m}{(l+\in)^n.(l+\in)^{-m}}$$

$$\sum_{n=1}^{\infty} u_n > \frac{u_m}{(l+\in)^{-m}} \sum_{n=1}^{\infty} \frac{1}{(l+\in)^n} \qquad \ldots(4)$$

Since $\sum_{n=1}^{\infty} u_n > \sum_{n=1}^{\infty} \frac{1}{(l+\in)^n}$ is a G.P series with c.r = $\frac{1}{(l+\in)} > 1$

Since $l+\in < 1 \quad \Rightarrow \quad \frac{1}{(l+\in)} > 1$

It follows by comparison test, series on L.H.S also diverges

Case III : When $l = 1$

(a) Let $\sum u_n = \sum \frac{1}{n^2}$

$$\frac{u_n}{u_{n+1}} = \frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}}$$

(a) Let $\sum u_n = \sum \frac{1}{n}$

Here $u_n = \frac{1}{n}$

$$u_{n+1} = \frac{1}{n+1}$$

$$= \frac{(n+1)^2}{n^2} = n^2 \frac{\left(1+\frac{1}{n}\right)^2}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 = 1$$

Here, in (a) part series in convergent.

By *p*–series test while in (b) part series in divergent.

But in both cases $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$

$\therefore$ Ratio test fails.

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}}$$

$$= \lim_{n\to\infty} \frac{n+1}{n}$$

$$= \lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n} = 1$$

Example 5.4.1 : Test the convergence of the following :

(i) $\frac{1}{3} + \frac{2!}{9} + \frac{3!}{27} + \frac{4!}{81} + ... \frac{n!}{3^n} + ...$

(ii) $\frac{1}{3} + \frac{x}{36} + \frac{x^2}{243} + ...$

Solution (i) $\frac{1}{3} + \frac{2!}{9} + \frac{3!}{27} + \frac{4!}{81} + ...$

Let $u_n = \frac{n!}{3^n} \quad \Rightarrow \quad u_{n+1} = \frac{(n+1)!}{3^{n+1}}$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{\frac{n!}{3^n}}{\frac{(n+1)!}{3^{n+1}}}$$

$$= \lim_{n \to \infty} = \frac{n!}{3^n} \cdot \frac{3^{n+1}}{(n+1)!}$$

$$= \lim_{n \to \infty} \frac{n! 3^n . 3}{3^n (n+1) n!}$$

$$= \lim_{n \to \infty} \frac{3}{n+1} \to \frac{3}{\infty} = 0 < 1$$

$\therefore$ by ratio test, series is divergent.

(ii) Let $u_n = \frac{x^{n-1}}{3^n n^2}$ ...(1)

$\Rightarrow \quad u_{n+1} = \frac{x^n}{3^{n+1} (n+1)^2}$

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \lim_{n \to \infty} \frac{\frac{x^{n-1}}{3^n n^2}}{\frac{x^n}{3^{n+1} (n+1)^2}}$$

$$= \lim_{n \to \infty} \frac{x^{n-2}}{3^n n^2} \cdot \frac{3^{n+1} (n+1)^2}{x^n} \to \frac{3}{x}$$

Case I : If $\frac{3}{x} > 1$ or 3 > x, then $\sum u_n$ is convergent.

Case II : If $\frac{3}{x} < 1$ < 1 or 3 < x, then $\sum u_n$ is divergent.

Case III : If $\frac{3}{x} = 1$ *i.e.* x = 3, ratio test fails

Put x = 3 in (1)

$$u_n = \frac{3^{n-1}}{3^n . n^2} = \frac{1}{3n^2}$$

$$\sum u_n = \sum \frac{1}{3n^2} = \frac{1}{3}\sum \frac{1}{n^2}$$

$\therefore$ by p–series test, it is convergent.

5.5. CAUCHY ROOT TEST

Statement An infinite series $\sum u_n$ be such that $\lim\limits_{n\to\infty} u_n^{\frac{1}{n}} = l$

(i) then $\sum u_n$ is convergent, if $l < 1$

(ii) $\sum u_n$ is divergent, If $l > 1$

(iii) Test fails if $l = 1$.

Proof : As $u_n^{\frac{1}{n}} \to l$, so for $\in > 0$, there exist m such that

$$\left| u_n^{\frac{1}{n}} - l \right| < \in, \quad \forall\, n \geq m \quad \Rightarrow \quad l - \in < u_n^{\frac{1}{n}} < l + \in, \ \forall\, n \geq m$$

Raise power 'n', we get

$$(l - \in)^n < u_n < (l + \in)^n, \quad \forall\, n \geq m \qquad ...(1)$$

Case I : When $l < 1$, clearly $l < 1 + \in < l$

From (1), $u_n \leq (l + \in)^n, \quad \forall\, n \geq m$

$$\sum_{n=1}^{\infty} u_n < \sum_{n=1}^{\infty} (1 + \in)^n \qquad ...(2)$$

But $\sum (1+\in)^n = (1+\in) + (1+\in)^2 + (1+\in)^3 + ...$

Is a G.P series with c.r. = $1 + \in$ (<1).

Hence, from (2), series an L.H.S $\sum u_n$ also converges

Case II : When $l > 1$, then

Clearly $1 < l - \in < l$ from (1)

$$(l - \in)^n < u_n, \quad \forall\, n \geq m$$

$$\sum_{n=1}^{\infty} (1 - \in)^n < \sum_{n=1}^{\infty} u_n$$

or $$\sum_{n=1}^{\infty} u_n > \sum_{n=1}^{\infty} (1 - \in)^n \qquad ...(3)$$

As R.H.S of (3) is a G.P series with c.r. $l - \in > 1$.

So is divergent. Hence, series an L.H,S of (3) also diverges.

Case III : When $l = 1$

(a) Consider $\sum u_n = \sum \frac{1}{n^2}$

Here, $u_n = \frac{1}{n^2}$

$$\Rightarrow \qquad u_n^{\frac{1}{n}} = \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \frac{1}{(n^2)^{\frac{1}{n}}}$$

$$u_n^{\frac{1}{n}} = \frac{1}{(n^{\frac{1}{n}})^2}$$

$$\lim_{n \to \infty} u_n^{\frac{1}{n}} = \lim_{n \to \infty} \frac{1}{(n)^{1/n}} \to \frac{1}{(1)^2} = 1$$

(b) Consider $\sum u_n = \frac{1}{n}$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = \frac{1}{1} = 1$$

Note that in (a) part series is convergent, by *p*-series test, while in (b) part series is divergent. But $\lim_{n\to\infty} u_n^{\frac{1}{n}} = 1$, which means root test fails i.e because for convergent and divergent. series $\lim_{n\to\infty} u_n^{\frac{1}{n}} = 1$.

Example 5.5.1. Test the following series :

(i) $\sum \frac{n^{n^2}}{(n+1)^{n^2}}$ (ii) $\sum \left(1 - \frac{1}{n}\right)^{n^2}$

Solution : (i) Let $\sum u_n = \sum \frac{n^{n^2}}{(n+1)^{n^2}}$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n}$$

$$= \lim_{n\to\infty} \left[\frac{\cancel{n}}{\cancel{n}\left(1+\frac{1}{n}\right)}\right]^{n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$= \frac{1}{e} = \frac{1}{2.3} < 1 \qquad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

$\Rightarrow \sum u_n$ is convergent.

(ii) Let $\sum u_n = \sum\left(1-\frac{1}{n}\right)^{n^2}$

Here $u_n = \left(1-\frac{1}{n}\right)^{n^2}$

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty}\left[\left(1-\frac{1}{n}\right)^{n^2}\right]^{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty}\left[1+\left(-\frac{1}{n}\right)^{-n}\right]^{-1}$$

$$= e^{-1} = \frac{1}{e} < 1$$

$\Rightarrow \sum u_n$ is convergent.

5.6. RAABE'S TEST

An infinite series $\sum u_n$ be such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = l$.

Then,

1. series is divergent, if $l < 1$
2. series is convergent, if $l > 1$
3. test fails for $l = 1$

Proof : It is given $n\left(\frac{u_n}{u_{n+1}}-1\right) \to l$, means, for $\in > 0, \exists m \in \mathrm{N}$

such that $\left| n\left(\dfrac{u_n}{u_{n+1}} - 1\right) - l \right| < \in, \ \forall\, n \geq m$

$$l - \in < n\left(\frac{u_n}{u_{n+1}} - 1\right) < l < \in, \ \forall\, n \geq m$$

$$l - \in < \left(\frac{n(u_n - u_{n+1})}{u_{n+1}}\right) < l < \in, \ \forall\, n \geq m$$

Multiply by u_{n+1} to whole

$$(u_{n+1})(l - \in) < [n(u_n - u_{n+1})] < (l + \in)u_{n+1}, \quad \forall\, n \geq m$$

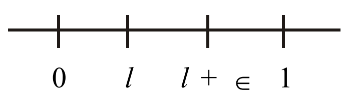
$$(l - \in)u_{n+1} < [nu_n - nu_{n+1})] < (l + \in)u_{n+1}, \quad \forall\, n \geq m$$

Add $-u_{n+1}$ to whole

$$(l - \in)u_{n+1} - u_{n+1} < [nu_n - nu_{n+1} - u_{n+1})] < (l + \in)u_{n+1} - u_{n+1}, \ \forall\, n \geq m$$

$$(l - \in -1)u_{n+1} < [nu_n - (n+1)u_{n+1}] < (l + \in -1)u_{n+1}, \ \forall\, n \geq m \qquad ...(1)$$

Case I When $l < 1$. Clearly, $l < l + \in -1 < 0$ ...(2)



Consider from (1)

$$nu_n - (n+1)u_{n+1} < (l + \in -1)u_{n+1}, \ \forall\, n \geq m$$

Put $n = m, m+1, ..., n-1$

$$\left.\begin{aligned}
mu_m - (m+1)u_{m+1} &< (l + \in -1)u_{m+1} \\
(m+1)u_{m+1} - (m+2)u_{m+2} &< (l + \in -1)u_{m+2} \\
(m+2)u_{m+2} - (m+3)u_{m+3} &< (l + \in -1)u_{m+3} \\
&: \\
&: \\
&: \\
(n-1)u_{n-1} - nu_2 &< (1 + \in -1)u_n
\end{aligned}\right\} \quad (n-1) - (m-1) = n - m$$

Add above $(n - m)$ inequalities

$$mu_m - nu_n < (l + \in -1)[u_{m+1} + u_{m+2} + ... + u_n] \qquad ...(3)$$

We know $S_n = u_1 + u_2 + ... + u_n$

$$= u_1 + u_2 + ... + u_m + ... + u_n$$

$$= u_1 + u_2 + ... + u_m + u_{m+1} .... + u_n$$

$$S_n = S_m + u_{m+1} + u_{m+2} + ... + u_n$$

$\Rightarrow \quad u_{m+1} + u_{m+2} + ... + u_n = S_n - S_m$

Use in (3), we get

$$mu_m - nu_n < (l + \in -1)(S_n - S_m)$$

$$(S_n - S_m)(l + \in -1) > mu_m - nu_n$$

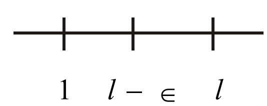
$$(S_n - S_m) > \frac{mu_m - nu_n}{l + \in -1}$$

$$S_n > S_m + \frac{mu_m - nu_n}{l + \in -1}$$

$$S_n > k, \text{ where } k = S_m + \frac{mu_m - nu_n}{l + \in -1}$$

$\Rightarrow \quad \{S_n\}$ of partial sum is bounded below thus, series $\sum u_n$ is divergent.

Case II : When $l > 1$. Then $1 < l - \in < l$



Consider, from (1)

$$(l - \in -1)\,u_{n+1} < [nu_n - (n+1)\,u_{n+1}] < (l + \in -1)u_{n+1}, \ \forall\, n \geq m$$

and proceed as case (1) ,we see that sequence $\{S_n\}$ of partial sum is bounded above, then series is convergent.

Case III : When $l = 1$.

Consider

(*a*) $\sum \frac{1^2.2^2......(2n-1)^2}{2^2.4^2........(2n)^2}$ a convergent Series

(*b*) $\sum \frac{1}{n}$ a divergent Series.

But in both the cases $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=1$.

Example 5.6.1. Test the convergence of the following series :

(1) $\sum \frac{1.3.5.....(2n-1)}{2.4.6......(2n)}.\frac{1}{n}$

(2) $\sum \frac{(n!)^2(x^n)}{(2n)!}$

(3) $\sum \frac{2.4.6.....(2n)}{1.3.5......(2n-1)}$

(4) $\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+...........$

(5) $x^2+\frac{2^2}{3.4}x^4+\frac{2^2 4^2}{3.4.5.6}x^6+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8..........$

(6) $\frac{1}{2}\times\frac{x^3}{2}+\frac{1.3}{2.4}\times\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\times\frac{x^7}{7}+..........$

Solution : (1) Let $\sum u_n = \sum \frac{1.3.5.....(2n-1)}{2.4.6......(2n)}.\frac{1}{n}$

$$u_n = \frac{1.3.5.....(2n-1)}{2.4.6......(2n)}.\frac{1}{n}$$

$$\Rightarrow \quad u_{n+1} = \frac{1.3........(2n-1)(2n+1)}{2.4.......(2n).(2n+2)}.\frac{1}{n+1}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)$$

$$= \lim_{n\to\infty} n\left[\frac{\frac{1.3.......(2n-1)}{2.4.......2n}\cdot\frac{1}{n}}{\frac{1.3.......(2n-1)(2n+1)}{2.4.......(2n).(2n+2)}\cdot\frac{1}{n+1}}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(1.3......2n-1)(2.4......2n)(2n+2)(n+1)}{(2.4.......2n).n(1.3......(2n-1)(2n+1)}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(2n+2)(n+1)}{n(2n+1)}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\not{2n^2}+2n+2n+2-\not{2n^2}-n}{\not{n}(2n+1)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{3n+2}{(2n+1)}\right]$$

Use L.H rule

$$= \frac{3}{2} > 1$$

$\Rightarrow \quad \sum u_n$ is convergent.

(2) Let $\sum u_n = \sum \frac{(n!)^2}{(2n)!}x^n$

$$u_n = \frac{(n!)^2}{(2n)!}x^n \qquad ....(1)$$

$$\Rightarrow \quad u_{n+1} = \frac{[(n+1)!)^2 x^{n+1}}{[2(n+1)]!} = \frac{[(n+1).n!]^2}{(2n+2)!}x^n.x$$

$$= \frac{(n+1)^2 (n!)^2}{(2n+2)(2n+1)(2n)!} x^n . x \text{, we use ratio test}$$

$$\frac{u_n}{u_{n+1}} = \frac{\frac{(n!)^2 x^n}{(2n)!}}{\frac{(n+1)^2 (n!)^2}{(2n+2)(2n+1)(2n)!} x^n . x}$$

$$= \frac{(n!)^2 x^n (2n+2)(2n+1)(2n)!}{(2n)! (n+1)^2 (n!)^2 x^2 . x}$$

$$= \frac{(2n+2)(2n+1)}{(n+1)^2 . x} = \frac{n^2 \left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{n^2 \left(1+\frac{1}{n}\right)^2 . x} = \frac{4}{x}$$

Case I : If $\frac{4}{x} > 1$ or $4 > x$ or $x < 4$, then $\sum u_n$ is convergent.

Case II : If $\frac{4}{x} < 1$ or 4 < x or $x > 4$, then $\sum u_n$ is divergent.

Case III : If $\frac{4}{x} < 1$ or $x = 4$. Then ratio test fails.

Put $x = 4$ in (1), $u_n = \frac{(n!)^2}{(2n)!} 4^n$

We get Raabe's test

$$\frac{u_n}{u_{n+1}} = \frac{\frac{(n!)^2}{(2n)!} 4^n}{\frac{[(n+1)!]^2}{[2(n+1)]!} 4^{n+1}}$$

$$= \frac{(n!)^2 4^n}{(2n)!} \frac{(2n+2)!}{[(n+1)\,n!]^2 4^{n+1}}$$

$$= \frac{\cancel{(n!)^2 4^n}}{\cancel{(2n)!}} \frac{(2n+2)(2n+1)\cancel{(2n)!}}{(n+1)^2 \cancel{(n!)^2 4^n}.4}$$

$$= \frac{(2n+2)(2n+1)}{4(n+1)^2}$$

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+2)(2n+1)}{4(n+1)^2} - 1$$

$$= \frac{4n^2 + 6n + 2 - 4(n^2 + 2n + 1)}{4(n+1)^2} = \frac{-2n-2}{4(n+1)^2}$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{-2n-2}{4(n+1)^2}\right) = \frac{n(n)\left(-2 - \frac{2}{n}\right)}{4n^2\left(1 + \frac{1}{n}\right)^2}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty}\left(\frac{-2 - \frac{2}{n}}{4\left(1 + \frac{1}{n}\right)^2}\right) = \frac{-1}{2} < 1$$

By Raabe's Test is divergent.

(6) $\frac{1}{2}.\frac{x^3}{3} + \frac{1}{2}.\frac{3}{4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + ...$

Let $\sum u_n = \frac{1.3.5......(2n-1)}{2.4.6......(2n)} \frac{x^{2n+1}}{(2n+1)}$ ...(1)

$$\Rightarrow \quad u_{n+1} = \frac{1.3.5......(2n-1)}{2.4.6......(2n+2)}\frac{x^{2n+3}}{(2n+3)} = \frac{1.3.5.....(2n-1)(2n+1)}{2.4.6......(2n)(2n+2)}\frac{x^{2n+3}}{(2n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{\dfrac{1.3.5.......(2n-1)\,x^{2n+1}}{2.4.6.......(2n)(2n+1)}}{\dfrac{1.3.5.........(2n-1)(2n+1)\,x^{2n+3}}{2.4.......(2n)(2n+2)(2n+3)}}$$

$$= \frac{1.3.5.......(2n-1)\,x^{2n+1}}{2.4.6.......(2n)(2n+1)} \cdot \frac{2.4.......(2n)(2n+2)(2n+3)}{1.3.5......(2n-1)(2n+1)\,x^{2n+1}x^2} = \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)\,x^2} \quad ...(1)$$

$$= \frac{2n\left(1+\frac{2}{2n}\right)2n\left(1+\frac{3}{2n}\right)}{2n\left(1+\frac{1}{2n}\right)2n\left(1+\frac{1}{2n}\right)x^2} = \frac{1}{x^2} \text{ as } n \to \infty$$

Case I : If $\frac{1}{x^2} > 1$, then $\sum u_n$ is convergent.

Case II : If $\frac{1}{x^2} < 1$, then $\sum u_n$ is divergent.

Case III : If $\frac{1}{x^2} = 1$ or $x^2 = 1$, Ratio test fails $\frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)(2n+1)}$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left[\frac{(2n+2)(2n+3)}{(2n+1)(2n+1)} - 1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{4n^2+6n+4n+6-4n^2-4n-1}{(2n+1)(2n+1)}\right]$$

$$= \lim_{n\to\infty} n\left(\frac{6n+5}{(2n+1)(2n+1)}\right) = \lim_{n\to\infty} \frac{\cancel{n}\,(6\cancel{n})\left(1+\frac{5}{6n}\right)}{2\cancel{n}\left(1+\frac{1}{2n}\right)2\cancel{n}\left(1-\frac{1}{2n}\right)}$$

$$= \frac{6}{4} = \frac{3}{2} > 1$$

By Raabe's Test $\sum u_n$ is convergent.

5.7. EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

Q.1. Test the following series :

(i) $\sum \frac{n^{n^2}}{(n+1)n^2}$ (ii) $\sum \left(1-\frac{1}{n}\right) n^2$

Q.2. Test the convergence of the following

(i) $\frac{1}{3}+\frac{2!}{9}+\frac{3!}{27}+\frac{4!}{81}+.........+\frac{n!}{3^n}+........$

(ii) $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+..........\frac{x^{2n-1}}{(2n-1)!}+..........$

Q.3. State and prove p-series test.

Q.4. State and prove ratio test

5.8. SUGGESTED READING

The students are advised to go through following references for details

5.9. REFERENCES

(1) Real analysis by by J.N. Kapur & H.C. Saxena, S.Chand & Co.

(2) Real analysis by J.N. Sharma & A.R.Vashishtha,Krishna Publication, New Delhi.

(3) Real analysis by Richard R. Goldberg, Oxford & IBH Publication Co. Pvt. Ltd. New Delhi.

(4) A text Book of Real Analysis by Sunil Gupta, Narinder Kumar, Malhotra Brothers Pacca Danga, Jammu.

(5) Real analysis by Jagdish Parsad Mittal & Neeraj Doda, Sharma Publication, Jallandhar.

5.10. MODEL TEST PAPER

Q.1. Test the following series :

(i) $\sum \frac{n^{n^2}}{(n+1)n^2}$ (ii) $\sum \left(1-\frac{1}{n}\right)^{n^2}$

(iii) $\frac{1}{3}+\frac{2!}{9}+\frac{3!}{27}+\frac{4!}{81}+.........\frac{n!}{3^n}+........$

(iv) $x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+.........\frac{x^{2n-1}}{(2n-1)}+.......$

Q.2. State and prove Ratio test.

Q.3. State and prove *p*-series test.

Q.4. State and prove Raabes test.

Q.5. Test the conversion of the following series :

(i) $\sum \frac{(n!)^2(x^n)}{(2n)!}$ (ii) $\sum \frac{1.3.5......(2n-1)}{2.4.6.......(2n)}.\frac{1}{n}$

(iii) $\sum \frac{2.4.6......(2n)}{1.3.5.......(2n-1)}$

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ALTERNATE SERIES

6.1. Introduction : In this lesson the concept of alternate series are discussed.

6.2 Objectives : Objective of studying this lesson is to explain concept of convergence of alternate & absolute series.

6.3. ALTERNATE SERIES

A series of the type

$$\sum_{n=1}^{\infty}(-1)^{n-1}u_n = u_1 - u_2 + u_3 - u_4 + \text{.........}, \text{each } u_n > 0$$

Is called an alternative series.

Example 6.3.1. (i) 1–3 + 5–7 +............. (ii) $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\text{...........}$

6.3.2. Lebnitz Test : An alternate series

$$\sum_{n=1}^{\infty}(-1)^{n-1}u_n = u_1 - u_2 + u_3 - u_4 + \text{.........} \text{ is such that}$$

(1) $u_1 > u_2 > u_3 > \text{........} > u_n > u_{n+1} > \text{.........} i.e. \{u_n\}$ is decreasing.

(2) $u_n \to 0$ as $n \to \infty$. Then $\sum(-1)^{n-1}u_n$ is convergent.

Proof : Let $\{S_n\}$ be sequence of partial sum of series $\sum(-1)^{n-1}u_n$

Consider $S_{2n} = u_1 - u_2 + u_3 - u_4 + \text{.........}u_{2n}$ ...(α)

Also, $S_{2n+1} = (u_1 - u_2 + u_3 - u_4 + ..., u_{2n}) + u_{2n+1}$

$S_{2n+1} = S_{2n} + u_{2n+1}$ ...(1)

$\Rightarrow \quad S_{2n+1} - S_{2n} > 0 \quad \therefore \quad u_{2n+1} > 0$

$S_{2n+1} \geq S_{2n} > 0$

or $\{S_{2n}\}$ is increasing sequence.

From (α)

$$S_{2n} = u_1 - (u_2 + u_3 - u_4 + ...u_{2n}) < u_1$$

$\Rightarrow \quad S_{2n} < u_1$

$\Rightarrow \quad \{S_{2n}\}$ is bounded above.

Thus $\{S_{2n}\}$ is increasing and bounded above, so by Monotone convergence theorem, $\{S_{2n}\}$ converges.

Let $S_{2n} \to S$ (say) *i.e.* $\lim_{n \to \infty} S_{2n} = S$

From (1)

$$\lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} u_{2n+1} = S + 0 \qquad \left[\because \lim_{n \to \infty} u_n = 0 \text{ by } (2)\right]$$

$$\lim_{n \to \infty} S_{2n+1} = S \quad \Rightarrow \quad S_{2n+1} \to S$$

Hence $S_{2n+1} \to S,\ S_{2n+2} \to S \quad \Rightarrow \quad S_n \to S$

i.e. Partial Sum $\{S_n\}$ of $\sum (-1)^{n-1} u_n$ also converges.

$\therefore \quad \sum (-1)^{n-1} u_n$ is convergence Series.

Example 6.3.3. Test the convergence of following :

(1) $1 - \frac{1}{\sqrt[2]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[4]{4}} + ...(-1)^{n-1} + ...$

(2) $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4}..............$

(3) $\frac{1}{\log 2}-\frac{1}{\log 3}+\frac{1}{\log 4}-\frac{1}{\log 5}+\ldots\ldots\ldots\ldots$

(4) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots\ldots\ldots+\frac{(-1)^{n-1}}{2n-1}$

(5) $2-4+6-8+\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution : (1) $1-\frac{1}{\sqrt[2]{2}}+\frac{1}{\sqrt[3]{3}}-\frac{1}{\sqrt[4]{4}}+\ldots\ldots\ldots=1-\frac{1}{(2)^{\frac{1}{2}}}+\frac{1}{(3)^{\frac{1}{3}}}-\frac{1}{(4)^{\frac{1}{4}}}$

(a) $u_1 > u_2 > u_3 > \ldots\ldots\ldots\ldots\ldots\ldots$

(b) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{(n)^{\frac{1}{n}}} = 1 \not\to 0$

Since 2nd condition of Lebnitz test fails, we can't apply Lebnitz test to this series

(2) $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\ldots\ldots\ldots\ldots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2^{n-1}}$

(a) Clearly $u_1 > u_2 > u_3 > \ldots\ldots\ldots\ldots\ldots\ldots$

(b) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{2^{n-1}} = 0$

As above series satisfies both conditions of Lebnitz test so given series is convergent

(3) $\frac{1}{\log 2}-\frac{1}{\log 3}+\frac{1}{\log 4}-\frac{1}{\log 5}+\ldots\ldots\ldots\ldots+(-1)^{n-1}\frac{1}{\log(n+1)}+\ldots\ldots\ldots\ldots$

Hence (a) $u_1 > u_2 > u_3 > \ldots\ldots$

(b) $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\log(n+1)} = 0$

As above series satisfies both conditions of lebnitz test so given series is convergent.

(4) Same as (i) Part.

(5) 2 – 4 + 6 – 8 +..........

u_n = 2 – 4 + 6 – 8 + ...(2n)............

$u_1 < u_2 < u_3 <$ ....means 1st condition of Lebnitz test fails, so we cant apply Lebnitz test to this series.

6.4. ABSOLUTELY CONVERGENT SERIES

Definition 6.4.1. An Alternate series

$$\sum_{n=1}^{\infty}(-1)^{n-1}u_n = u_1 - u_2 + u_3 - u_4 t............$$

is said to be absolutely convergent series if series $\left|\sum(-1)^{n-1}u_n\right|$

$$= \sum_{n=1}^{\infty}|(-1)^{n-1}u_2| = \sum_{n=1}^{\infty}|u_n| \text{ converges.}$$

Example 6.4.2. Prove that every absolutely convergent series is convergent but not conversely.

Solution : Let $\sum(-1)^{n-1}u_n$ be an absolutely convergent series.

This means $|\sum(-1)^{n-1}u_n|$ is convergent.

or $$\sum_{n=1}^{\infty} |\sum(-1)^{n-1}u_n| = \sum_{n=1}^{\infty}|u_1| + |u_2| + |u_3| +....... \text{ is convergent.}$$

Now $$\sum_{n=1}^{\infty}|(-1)^{n-1}u_n| = |u_1 - u_2 + u_3 - u_4 + ...|$$

$$\leq |u_1| + |-u_2| + |u_3| +........$$

$$= \sum_{n=1}^{\infty}|u_n|$$

Since R.H.S $\sum_{n=1}^{\infty} |u_n|$ is convergent so in L.H.S

Thus series $\left|\sum (-1)^{n-1} u_n\right|$ is convergent.

Converse of above result need not to be true.

6.4.3. Example of a convergent series which is not absolutely convergent.

Let $\sum (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots\ldots$

Then by Lebnitz test, $\sum (-1)^{n-1} \frac{1}{n}$ is convergent.

But $\left|\sum (-1)^{n-1} \frac{1}{n}\right| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots\ldots$

$= \sum \frac{1}{n}$ is divergent by p–sries test.

Remark : A series which converges but not absolutely is called conditional convergent series.

6.5. EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

Q. 1. Test the convergence of following :

(1) $1 - \frac{1}{\sqrt[2]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[4]{4}} + \ldots\ldots\ldots\ldots(-1)^{n-1} + \ldots\ldots\ldots$

(2) $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} \ldots\ldots\ldots\ldots$

(3) $\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots\ldots\ldots\ldots$

(4) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\ldots\ldots + \frac{(-1)^{n-1}}{2n-1}$

(5) 2 – 4 + 6 – 8 +......................

6.6. SUGGESTED READING

The students are advised to go through following references for details

6.7. REFERENCES

(1) Real analysis by by J.N. Kapur & H.C. Saxena, S.Chand & Co.

(2) Real analysis by J.N. Sharma & A.R.Vashishtha,Krishna Publication, New Delhi.

(3) Real analysis by Richard R. Goldberg, Oxford & IBH Publication Co. Pvt. Ltd. New Delhi.

(4) A text Book of Real Analysis by Sunil Gupta, Narinder Kumar, Malhotra Brothers Pacca Danga, Jammu.

(5) Real analysis by Jagdish Parsad Mittal & Neeraj Doda, Sharma Publication, Jallandhar.

6.8. MODEL TEST PAPER

Q.1. Prove that every absolutely convergent series is convergent but not conversely.

Q.2. State and prove Lebenitz test.

Q.3. Test the convergence of following :

(i) $1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\ldots\ldots\ldots\ldots\frac{(-1)^{n-1}}{\sqrt[n]{n}}+\ldots\ldots\ldots$

(ii) $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}\ldots\ldots\ldots\ldots$

(iii) $\frac{1}{\log 2}-\frac{1}{\log 3}+\frac{1}{\log 4}-\frac{1}{\log 5}+\ldots\ldots\ldots\ldots$

Q.4. Give an example of a convergent series which is not absolutely convergent.

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CONTINUOUS FUNCTIONS

7.1. Introduction : In this lesson the concept of continuity of functions are discussed. The concept is explained in a simpler way.

7.2. Objectives : Objective of studying this lesson is to give the idea of continuity of functions both in algebraic & graphical forms.

7.3. INTRODUCTION

First we shall introduce the concept of limit of a function whose domain is an interval and whose range is contained in **R**.

7.3.1. Definition of Limit : A number l is said to be the limit of $f(x)$ at $x = a$ iff for any arbitrarily chosen positive number , however small but not zero, there exists a corresponding number greater than zero such that

$$| f(x) - l | < \in$$

for all values of x for which $0 < | x - a | < \delta$.

Meaning of $| x - a | < \delta$.

Since $| x - a |$ means the absolute value of $x - a$ without regard to sign, the inequality $| x - a | < \delta$, means that the difference between x and a taken positively, is less than δ. Thus

(i) if $x > a$, then $x - a < \delta$.

(ii) if $x < a$, then $a - x < \delta$.

In other words, if $x > a$, then $x < a + \delta$ and if $x < a$, then $x > a - \delta$.

Hence $| x - a | < \delta$ means that x can be assigned any value between $a - \delta$ and $a + \delta$.

Right hand and left hand limits.

If x approaches a from right, that is, from values of x greater than a, the limit of *f* as defined above is called the right hand limit on *f (x)* and is written as :

$$\lim_{x \to a+0} f(x) \quad \text{or} \quad f(a+0) \quad \text{or} \quad \lim_{x \to a^+} f(x)$$

Formally we may define Right hand limit as under :

"A function *f (x)* is said totend to a limit *l* through right hand if for any arbitrarily chosen positive number *e* however small, but not zero, there exists a corresponding $\delta > 0$ such that

$$| f(x) - l | < \in$$

for all values of *x* such that $a < x < a + \delta$".

The working rule for finding the right hand limit is : Put $x = a + h$ in f (x) and make h approach zero.

Similarly if x approaches a from the left, that is, from values of *x* smaller than *a*, the limit of *f* in that case is called the Left hand limit and is written as

$$\lim_{x \to a-0} f(x) \quad \text{or} \quad f(a-0) \quad \text{or} \quad \lim_{x \to a^+} f(x)$$

formally we may define Left hand limit as under :

A function *f (x)* is said to tend to limit *l* through left hand iff for any arbitrarily chosen positive number however small but not zero, there exists a corresponding number $\delta > 0$ such that $a - \delta < x < a$.

Remark 7.3.2. The limit of the function *f (x)* is said to exist if both right hand and left hand limits exist and are equal *i.e.*

$$\lim_{x \to a-0} f(x) = \lim_{x \to a+0} f(x) = l$$

The common value is called the Limit of the function and is written as :

$$\lim_{x \to a+0} f(x) = l$$

(2) In case of Left hand limit is not equal to the right hand limit, the limit of the function does not exist. Also the limit of the function does not exist if either one both of these limits donot exist.

EXAMPLE 7.3.3. (1) Let a function *f* be defined as

$f(x) = -1$ when $x < 0$

$= 0$ when $x = 0$

$= 1$ when $x > 0$

Then $\lim\limits_{x \to 0-0} f(x) = -1$ and $\lim\limits_{x \to 0+0} f(x) = 1$

Here $\lim\limits_{x \to 0-0} f(x) \neq \lim\limits_{x \to 0+0} f(x)$

$\lim\limits_{x \to 0} f(x)$ does not exist.

(2) Let a function f be defined as

$$f(x) = \begin{cases} 1 - 2x & \text{when } x < 0 \\ 0 & \text{when } x = 0 \\ 1 + 3x & \text{when } x > 0 \end{cases}$$

Then $\lim\limits_{x \to 0-0} f(x) = \lim\limits_{x \to 0-0} (1 - 2x) = 1$

$\lim\limits_{x \to 0+0} f(x) = \lim\limits_{x \to 0+0} (1 + 3x) = 1$

Here $\lim\limits_{x \to 0} f(x) = 1$.

7.3.4. Algebra of Limit : Let f and g be two functions with a common domain D and whose ranges are in R.

The sum of the function f and g is the function $f + g$ defined on D by setting

$(f + g)(x) = f(x) + g(x)$ for all $x \in$ D.

Also, the product of the functions f and g is the function fg define on D by setting

$(fg)(x) = f(x) \cdot g(x)$, for all $x \in$ D.

Again, if c be any real number, the scalar product off by c is the function cf defined by setting $(cf)(x) = cf(x)$, for all $x \in$ D.

Further, if $g(x) \neq 0$ whenever $x \in D_1$, then the reciprocal of g is the function $\frac{1}{g}$ defined on D_1 be setting $\left(\frac{1}{g}\right)(x) = \frac{1}{g(x)}$, for all $x \in D_1$.

Finally, if $g(x) \neq 0$ whenever $x \in D_1, \subset D$, then the quotientis $\frac{f}{g}$ the function defined on D_1 by setting $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, for all $x \in D$.

We shall now study the relation between the limits of two functions and the limits of their sum, product etc.

Theorem 7.3.5. The limit of a sum is equal to the sum of the limits.

Proof. Let us assume, and $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} \varphi(x) = m$.

Then we have to prove that $\lim_{x \to \infty} [f(x) + \varphi(x)] = l + m$.

We have only to show that for any preassigned positive number $\in$, a number can be determined such that

$$| f(x) + \varphi(x) - l - m | < \in$$

whenever x lies in the interval $[a - \delta, a + \delta]$.

Now by hypothesis lim $f(x) = l$ so that

$$| f(x) - l | < \frac{\in}{2}, \text{ whenever } 0 < | x - a | < \delta_1 \qquad ...(1)$$

Similarly, $| \varphi(x) - m | < \frac{\in}{2}$ whenever $0 < | x - a | < \delta_2$ ...(2)

Choosing δ to be smaller of the number $\delta_1$and δ_2, it follows from (1) and (2) that

$$\begin{aligned} | f(x) + \varphi(x) - l - m | &= | f(x) - l + \varphi(x) - m | \\ &\leq | f(x) - l | + | \varphi(x) - m | \\ &< \frac{\in}{2} + \frac{\in}{2} = \in \end{aligned}$$

When $0 < | x - a | < \delta$

Hence $\lim_{x \to a} [f(x) + \varphi(x)] = l + m$

the same way we can prove that $\lim_{x \to a} [f(x) + \varphi(x)] = l - m$.

Theorem 7.3.6. The limit of a product is equal to the product of the limits.

Proof : Using the notation of theorem I, we have to prove in this case that

$$|f(x).\varphi(x) - lm| < \in \text{ whenever } 0 < |x - a| < \delta.$$

Now $\quad |f(x).\varphi(x) - lm| = |f(x).\varphi(x) - l\varphi(x) + l\phi(x) - lm|$

$$\leq |f(x).\varphi(x) - l\varphi(x)| + |l\varphi(x) - lm|$$

$$= |\varphi(x)||f(x) - l| + |l||\varphi(x) - lm|$$

By hypothesis $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} \varphi(x) = m$ in so φ that is surely bounded in the neighborhood of $x = a$.

Hence $|\varphi(x)| <$ M for all value of x such that $0 < |x - a| < \delta$.

Then $|f(x) \ \varphi(x) - \text{lm}| < \text{M}\,|f(x) - l| + |l|\,|\varphi(x) - m|$.

Since $\lim_{x \to a} f(x) = l$ and $\varphi(x) \to m$, coressponding to any $\in > 0$, we can find a positive number < such that $|f(x) - l| < \frac{\in}{2\text{M}}$ and $|\varphi(x) - m| < \frac{\in}{2|l|}$ whenever $0 < |x - a| < \delta$.

Hence $f(x)\varphi(x) \to lm$.

Theorem 7.3.7. The limit of quotient is equal to the quotient of the limits provided the limit of the denominator is not zero.

Proof : Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} \varphi(x) = m \neq 0$

Now $\quad \left|\frac{f(x)}{\varphi(x)} - \frac{l}{m}\right| \leq \left|\frac{f(x)}{\varphi(x)} - \frac{f(x)}{m}\right| + \left|\frac{f(x)}{m} - \frac{l}{m}\right|$

$$= \frac{|f(x)|}{|m||\varphi(x)|} \,.\, \{m - \varphi(x)\} + \frac{1}{m}\{f(x) - l\}$$

$$= \frac{|f(x)|}{|m||\varphi(x)|}|m - \varphi(x)| + \frac{1}{|m|}|f(x) - l| \qquad ...(1)$$

By hypothesis $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} \varphi(x) = m.$ Hence the functions f and φ are surely bounded in the neighborhood of the point $x = a$. Let M be the upper bounded of $|f|$ and N be the lower bounded of $|\varphi|$ so that $|f(x)| < M$ and $|\varphi(x)| > N$.

We may then write (1) as

$$\left| \frac{f(x)}{\varphi(x)} - \frac{l}{m} \right| \leq \frac{M}{N|m|}|m - \varphi(x)| + \frac{1}{|m|}|f(x) - l| \qquad ...(2)$$

Since $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} \varphi(x) = m,$ corresponding, to any $\in > 0$, we can find number δ_1 and δ_2 such that

$$|f(x) - l| < |m| \left(\frac{\in}{2} \right) \text{ whenever } 0 < |x - a| < \delta_1$$

and $$|\varphi(x) - m| < \frac{N}{M}|m| . \frac{\in}{2} \text{ whenever } 0 < |x - a| < \delta_2$$

Choosing δ to be smaller than δ_1 and δ_2, we see from (2) that

$$\left| \frac{f(x)}{\varphi(x)} - \frac{l}{m} \right| < \frac{\in}{2} + \frac{\in}{2} = \in \text{ whenever } 0 < |x - a| < \delta .$$

Hence $\lim_{x \to a} \frac{f(x)}{\varphi(x)} = \frac{1}{m}$, provided $m \neq 0..$

7.4. SOME IMPORTANT LIMITS

The following limits should be committed to memory by the students.

(A) $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, when φ is measured in radians.

(B) $\lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^x = e$ and $\lim_{y \to 0} (1 + y)^{\frac{1}{y}} = e$

(C) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

(D) $\lim_{x \to c} \frac{x^n - c^n}{x - c} = nc^{n-1}$

EXAMPLE 7.4.1. The function f defined on [0, 1] by

$$f(x) = \frac{1}{x},\ x \in]0, 1]$$

is continuous on]0, 1].

Solution. Let $c \in]0, 1]$ be arbitrary. Take $\delta_1 = \frac{c}{2} > 0$ such that

$$|x - c| < \delta_1 = \frac{c}{2} \quad \Rightarrow \quad \frac{c}{2} < x < \frac{3c}{2}$$

This gives $(1) \frac{2}{3c^2} < \frac{1}{cx} < \frac{2}{c^2}$

Let $\in > 0$ be given.

Then $|f(x) - f(c)| = \left| \frac{1}{x} - \frac{1}{c} \right| = \frac{|x - c|}{cx}$

$$< \frac{2}{c^2} |x - c|$$

$$< \in \text{ if } |x - c| < \frac{c^2}{2} \in$$

If we choose $\delta = \min \left\{ \frac{c^2}{2}, \in \right\}$

Then, we have $|f(x) - f(c)| < \in$ whenever $|x - c| < \delta$.

Hence f is continuous at c. Since $c \in [0, 1]$ is arbitrary, it follows that f is continuous on [0, 1].

7.5. DISCONTINUITY CRITERION

Let f be a real valued function defined on $I \subseteq R$ and $c \in I$. Then f is discontinuous at c if and only if there exists a sequence $<X_m>$ in l with, $\lim_{x \to c} x_n = c$ such that $\lim_{x \to c} f(x_n) \neq f(c)$.

7.5.1. Kinds of Discontinuities : (1) A function f is said to have a **removable** discontinuity at a point a iff $\lim_{x \to a} f(x)$ exists but is not equal to $f(a)$, *i.e.*, iff

$$f(a + 0) = f(a - 0) \neq f(a).$$

In such a case the function may be made continuous by defining it in such a way that

$$f(a) = \lim_{x \to a} f(x)$$

(2) If $f(a + 0)$ and $f(a - 0)$ both exist and not equal, then we say that it has a of the first kind at a.The point a is said to be a point of discontinuity from the left or right as $f(a-0) \neq f(a) = f(a+0)$ or $f(a-0) = f(a) \neq f(a+0)$.

(3) A function f is said to have a discontinuity of second kind at a iff none of $f(a + 0)$ and $f(a - 0)$ exists.

A point a is said to be a discontinuity of the second kind from the left or right according as $f(a + 0)$ and $f(a - 0)$ exists.

Example7.5.2. Test the continuity of the function $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$.

Solution. Here $f(0 + 0) = \lim_{h \to 0} (0 + h) \sin \frac{1}{0 + h}$

$= h \sin \frac{1}{h} = 0 \times$ a finite quantity

$= 0$ $\qquad$ [$\because$ $\sin \left(\frac{1}{h}\right)$ is bounded lying between -1 and 1]

Similarly $f(0-0) = \lim_{h \to 0} (0-h)\sin\frac{1}{0-h}$

$$= \lim_{h \to 0} (0-h)\sin\frac{1}{0-h} = \lim_{h \to 0} h\sin\frac{1}{h} = 0, \text{ as before.}$$

Also $f(0) = 0$.

Since $f(0+0) = f(0-0) = f(0)$, the function $x \sin \frac{1}{x}$ is continuous at $x = 0$.

Example7.5.3. Show that the function defined as

$$\varphi(x) = \begin{cases} 0, & \text{for } x \le 0 \\ \frac{1}{2} - x, & \text{for } 0 < x < \frac{1}{2} \\ \frac{3}{2} - x, & \text{for } \frac{1}{2} \le x < 1 \\ 1, & \text{for } x = 1 \end{cases}$$

has three points of discontinuity which you are required to find.

Solution. We test the function for continuity at $x = 0$, $\frac{1}{2}$ and 1.

For $x = 0$, we have $\varphi(0) = 0$, $\varphi(0+0) = \lim_{h \to 0}\left[\frac{1}{2} - (0+h)\right] = \frac{1}{2}$

Since $\varphi(0) = \varphi(0+0)$, the function is discontinuous at $x = 0$.

For $x = \frac{1}{2}$, we have

$$\varphi\left(\frac{1}{2}\right) = \varphi\left(\frac{1}{2} - 0\right) = \lim_{h \to 0}\left[\frac{1}{2} - \left(\frac{1}{2} - h\right)\right] = 0$$

$$\varphi\left(\frac{1}{2} + 0\right) = \lim_{h \to 0}\left[\frac{3}{2} - \left(\frac{1}{2} + h\right)\right] = 1$$

Since $\varphi\left(\frac{1}{2}-0\right) \neq \varphi\left(\frac{1}{2}\right) \neq \varphi\left(\frac{1}{2}+0\right)$, the function is discontinuous at $x = \varphi(1)\frac{1}{2}$.

Finally, we consider $x = 1$. We have

$$\varphi(1) = 1,\ \varphi(1-0) = \lim_{h\to 0}\left[\frac{3}{2}-(1-h)\right] = \frac{1}{2}$$

Since $\varphi(1-0) \neq \varphi(1)$ so the function is discontinuous at $x = 1$.

Hence the function is discontinuous at $x = 0, \frac{1}{2}$, and 1.

7.6. EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

Q.1. Let a function f be defined as

$$F(x) = \begin{cases} 1-2x \text{ when } x < 0 \\ 0 \quad \text{when } x = 0 \\ 1+3x \text{ when } x > 0 \end{cases}$$

Is F continuous function.

Q.2. Prove that sum of two continuous functions is continuous.

Q.3. Prove that product of two continuous functions is continuous.

7.7. SUGGESTED READING

The students are advised to go through following references for details.

7.8. REFERENCES

(1) Real analysis by by J.N. Kapur & H.C. Saxena, S.Chand & Co.

(2) Real analysis by J.N. Sharma & A.R.Vashishtha,Krishna Publication, New Delhi.

(3) Real analysis by Richard R. Goldberg, Oxford & IBH Publication Co. Pvt. Ltd. New Delhi.

(4) A text Book of Real Analysis by Sunil Gupta, Narinder Kumar, Malhotra Brothers Pacca Danga, Jammu.

(5) Real analysis by Jagdish Parsad Mittal & Neeraj Doda, Sharma Publication, Jallandhar.

7.9. MODEL TEST PAPER

Q.1. Prove that every continuous function is bounded.

Q.2. Give an example to show that a bounded function may not be continuous.

Q.3. Prove that sum of two continuous functions is continuous.

Q.4. Prove that quotient of two continuous functions is continuous.

Q.5. Show that the function defined as $\varphi(x) = \begin{cases} 0 & \text{for } x \le 0 \\ \frac{1}{2} - x & \text{for } 0 \le x < \frac{1}{2} \\ \frac{3}{2} - x & \text{for } \frac{1}{2} \le x < 1 \\ 1 & \text{for } x = 1 \end{cases}$

has three points of discontinuity.

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THEOREMS ON CONTINUITY

8.1. Introduction : In this lesson the properties of continuity of functions are discussed in the form of theorems.

8.2 Objectives : Objective of studying this lesson is to explain continuity in different approach in the form of results.

8.3. THEOREMS ON CONTINUITY

Theorem 8.3.1. The necessary and sufficient condition for a function f defined on $I \subset R$ to be continuous at $a \in I$ is that for each sequence $<a_n>$ which converges a, we have

$$\lim_{n \to \infty} f(a_n) = f(a).$$

Proof : Let f be continuous at $a \in I$ and let $<a_n>$ be a sequence such that

$$\lim_{n \to \infty} a_n = a$$

Since f is continuous at a, for given $\in > 0$, we can find $\delta > 0$ such that

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - f(a)| < \in \qquad ...(1)$$

Again since $\lim_{n \to \infty} a_n = a$, there exists a positive integer m such that

$$n > m \quad \Rightarrow \quad |a_n - a| < \delta \qquad ...(2)$$

Setting $x = a_n$ in(1), we get

$$|a_n - a| < \delta \quad \Rightarrow \quad |f(a_n) - f(a)| < \in$$

From (2) and (3), we get

$$n > m \quad \Rightarrow \quad |f(a_n) - f(a)| < \in$$

Hence $\lim_{n \to \infty} f(a_n) = f(a)$

Conversely, suppose for every sequence $<a_n>$ converging to a, we have

$$\lim_{n \to \infty} f(a_n) = f(a)$$

Then we shall show that f is continuous at a. If possible, let f be not continuous at a. Then there exists $\delta > 0$ such that for every $\delta > 0$ there is an $\in$ such that

$$| x - a | < \delta \text{ but } | f(x) - f(a) | \geq \in$$

If we take $\delta = \frac{1}{n}$, we see that for each positive integer n, there exists $\{a_n\}$ such that

$$| a_n - a | < \frac{1}{n} \text{ but } | f(a_n) - f(a) | \geq \in$$

Then $\lim_{n \to \infty} a_n = a$ but $\lim_{n \to \infty} f(a_n) \neq f(a)$

But this is a contradiction.

Hence function must be continuous at $x = a$.

8.3.2.Definition (Bounded). If the range of a function f is a bounded set, that is if both upper and lower bounds of the function exists and are finite, then the function is said to be bounded.

Equivalently, if there exists a number M > 0 such that $| f(x) | < M$ for all x, then f is said to be a bounded function.

Theorem 8.3.3. If f is continuous in the closed interval $[a, b]$, then

(1) f is bounded in $[a, b]$

(2) f attains its supremum and infimum at least once in $[a, b]$.

Proof : (1) Since f is continuous in $[a, b]$ so, for a given $\in > 0$, we can subdivide the interval into a finite number n of sub–intervals such that

$$| f(x_1) - f(x_2) | < \in \quad ...(1)$$

for any two points x_1, x_2 in the same sub–interval. Let x be any point in the first sub interval $[a, a_1]$. Then by (1) we have

$| f(a) - f(x_2) | < \in$

that is, $f(x)$ lies in the interval $f(a) - \in$ and $f(a) + \in$. In the same way, all the values $f(x)$ in the first two sub–intervals will lie between $f(a) - 2\in$ and $f(a) + 2\in$, and so on.Hence all the values of $f(x)$ in the interval $[a, b]$ will lie between $f(a) - n\in$ and $f(a) + n\in$. Thus f is bounded in $[a, b]$.

Note 8.3.4. The converse of the above result is not true, *i.e.* a bounded function in $[a, b]$ need not be continuous in $[a, b]$. For example, the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \textit{for } x \neq 0 \\ 0 & \textit{for } x = 0 \end{cases}$$

is bounded in [0, 1] but not continuous in [0, 1], since it is discontinuous at $x = 0$

Proof (ii) : Let M be m be the supremum and infimum of f in $[a, b]$ respectively. We shall show that f attains its supremum M at least once in this interval, *i.e.* there exists a point x in $[a, b]$ such that $f(x) = M$. Suppose it does not, then $M \neq f(x)$ or $M - f(x) \neq 0$ for any x in $[a, b]$.

Let us define a function g on $[a, b]$ by setting

$$g(x) = \frac{1}{M - f(x)} \text{ for all } x \in [a, b].$$

Since f is continuous on $[a, b]$, therefore, g is also continuous on $[a, b]$. As every continuous function defined on a closed interval is bounded, therefore, there exists a positive real number k such that $g(x) \leq k$ for all $x \in [a, b]$. [It means k is an upper bound of g) *i.e.*, $g(x)$ for all $x \in [a, b]$.

This means that $f(x) \leq M - \frac{1}{k}$ for all x $[a, b]$, so that $M - \frac{1}{k}$ is an upper bound o f $f(x)$. This contradict the fact that M is the supremum of f, and consequently there must exists some x in $[a, b]$ such that $M - f(x) = 0$.

Hence $f(x) = M$ for atleast one value of x in $[a, b]$. Similarly it can be proved that f attains its infimum at least once in $[a, b]$.

8.4. UNIFORM CONTINUITY

Recall the definition of continuity where f depends not only on $\in$ but also on the point catwhich the continuity is defined. Now $\in$ depends on the point c means that the change in the values of the function near some point may be different from other points.

Definition 8.4.1. A function f defined on an interval $I \subseteq \mathbf{R}$ is said to be uniformly continuous on l if for each $\in > 0$ there exists a $\delta = \delta(\in) > 0$ such that $| f(x) - f(y) | < \in$, whenever $| x - y | < \delta$ and $x, y \in \mathbf{I}$.

Examples 8.4.2. Consider the function $f(x) = x^2$, $x \in [-1, 1]$.

Solution. Let $x, y \in [-1, 1]$ be any two points.

Then $| f(x) - f(y) | = | x^2 - y^2 | = | x - y | | x + y | \leq 2 | x - y |$

$$(\because \; x, y \in [-1, 1] \;\Rightarrow\; | x | \leq 1 \text{ and } | y | \leq 1)$$

$$\Rightarrow \quad | f(x) - f(y) | < \in, \text{ if } | x - y | < \frac{\in}{2}$$

Thus, for any $\in > 0$ there exists $\delta = \frac{\in}{2} > 0$ such that

$$| f(x) - f(y) | < \in, \text{ whenever } | x - y | < \delta.$$

Hence f is uniformly continuous on $[-1, 1]$.

Example 8.4.3. Consider the function $f(x) = \sin x$, $x \in [0, +\infty]$.

Solution. Let $x, y \in [0, +\infty]$ be any two points. Then

$$| f(x) - f(y) | = | \sin x - \sin y |$$

$$= \left| 2 \sin \frac{x-y}{2} \cos \frac{x+y}{2} \right|$$

$$= \left| \sin \frac{x-y}{2} \right| \left| \cos \frac{x+y}{2} \right|$$

$$\leq 2 \left| \sin \frac{x-y}{2} \right| \qquad (\because \; | \cos \theta | \leq 1)$$

$$\leq | x - y | \qquad (\because \; | \sin x | \leq | \theta |)$$

Therefore, for any $\in > 0$, there exists a $\delta > 0$ such that

$| f(x) - f(y) | < \in$, whenever $| x - y | < \delta \ (= \in)$

Hence, f is uniformly continuous on $[0, +\infty]$.

8.5. NON–UNIFORM CONTINUITY CRITERION

A function f defined on an interval $I \subseteq R$ is not uniformly continuous on l if and only if there exists an $\in > 0$ such that for all $\delta > 0$ there are points x, y (depending on δ) in I such that $| x - y | < \delta$ and $| f(x) - f(y) \geq \in$

Example 8.5.1. Let f be a function defined on $]0, 1]$ by $f(x) = \dfrac{1}{x}$. Then

(a) f is continuous on $[0, 1]$.

(b) f is not uniformly continuous on $[0, 1]$.

Solution. Let $\delta > 0$ be any real number. Then by Archimedean Property, there exista positive integer m such that $\dfrac{1}{m} < \delta$.

Put $x = \dfrac{1}{m}$ and $y = \dfrac{1}{m+1}$. Then $x, y, \in [0, 1]$ such that

$$| x - y | = \left| \frac{1}{m} - \frac{1}{m+1} \right| = \frac{1}{m(m+1)} < \frac{1}{m} < \delta$$

and $$| f(x) - f(y) | = \left| \frac{1}{x} - \frac{1}{y} \right| = 1 > \in, \text{ for any } \in < 1.$$

Therefore, f is not uniformly continuous.

Theorem 8.5.2. A uniformly continuous function f defined on an interval $I \subseteq R$ is continuous on I.

Proof. Let f be uniformly continuous on I. Then, for each $\in > 0$, there exists $\delta > 0$ such that for $x, y \in I$.

(1) $| f(x) - f(y) | < \in$, whenever $| x - y | < \delta$.

Let $c \in I$ be any point. Since I is an interval, every sequence in I converging to c is either monotone increasing or monotone decreasing. Let $<x_n>$ be any monotone sequence in I such that $\lim_{n \to \infty} x_n = c$.

Then, for each $\delta > 0$, there exists a positive interger m such that

$$| x_n - c | < \delta, \quad \forall\, n \geq m$$

$$\Rightarrow \quad | f(x_n) - f(c) | < \in, \ \forall\, n \geq m$$

$$\Rightarrow \quad \lim_{n \to \infty} f(x_n) = f(c)$$

$\Rightarrow$ f is continuous at c.

Since $c \in$ I is any point, it follows that f is continuous on I.

Remark 8.5.3. A continuous function is not necessarily uniformly continuous. Indeed the continuous function f defined on R by $f(x) = x^2$ is not uniformly continuous since for any $\delta > 0$ there exists (by Archimedian Property) a positive integer m such that $\frac{1}{m} < \delta$.

Take $x = m$ and $y = m + \frac{1}{m}$..

Then $x, y \in \mathbf{R}$ such that $| x - y | = \frac{1}{m} < \delta$ and

$$| f(y) - f(x) | = \left| \left(m + \frac{1}{m} \right)^2 - m^2 \right| = \frac{1}{m^2} + 2 > 2 = \in.$$

Theorem 8.5.4. A continuous function f on a bounded closed interval $[a, b]$ is uniformly continuous.

Proof : Suppose f is not uniformly continuous on $[a, b]$. Then there exists an $\in_0$ > 0 such that b for all $\delta\left(= \frac{1}{n} \right) > 0,\ n \in \mathrm{N}$. There are points $x_n, y_n \in [a, b]$ such that

(1) $| x_n - y_n | < \frac{1}{n}$ and $| f(x_n) - f(y_n) | \geq \in_0$

We thus get sequence $<x_n>$ and $<y_n>$ in $[a, b]$ satisfying (1). Now $<x_n>$ is a bounded sequence, so $<x_n>$ has a convergent subsequence say $<x_{n_k}>$. Let $\lim_{k \to \infty} x_{n_k} = x.$ .Then $x \in [a, b]$, since $[a, b]$ is closed. Let $<y_{n_k}>$ be a subsequence of $<y_n>$. Then (1) gives.

(2) $|x_{n_k} - y_{n_k}| < \frac{1}{n_k}$ and $|f(x_{n_k}) - f(y_{n_k})| \geq \in_0$

note that $|y_{n_k} - x| \leq |y_{n_k} - x_{n_k}| = |x_{n_k} - x|$

$$< \frac{1}{n_k} + |x_{n_k} - x| \to 0, \text{ as } k \to \infty \qquad (\because \lim_{k\to\infty} x_{n_k} = 1)$$

$\Rightarrow \quad \lim y_{n_k} = x$

Now, since f is continuous at x and $\lim_{k\to\infty} x_{n_k} = x$, $\lim_{k\to\infty} f(x_{n_k}) = f(x)$

then, for each $\varepsilon > 0$, there exists a positive integer m such that

(3) $|f(x_{n_k}) - f(x)| < \in, \ \forall\, n \geq m$

Therefore $|f(y_{n_k}) - f(x)| = |f(y_{n_k}) - f(x_{n_k}) + f(x_{n_k}) - f(x)|$

$$= |f(y_{n_k}) - f(x_{n_k}) - (f(x) - f(x_{n_k}))|$$

$$\geq |f(y_{n_k}) - f(x_{n_k}) - (f(x) - f(x_{n_k}))|$$

$$\geq \in_0 - \in, \ \forall\, n \geq m \qquad \text{(by (2) and (3))}$$

$\Rightarrow \quad < f(y_{n_k}) >$ does not converges to $f(x)$. However $\lim_{k\to\infty} y_{n_k} = x$. This contradicts the facts that f is continuous at $x \in [a, b]$. Hence f must be uniformly continuous.

8.6. EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

1. Do the following limits exist ? If they exist, find their values:

(i) $\lim_{x\to a} \frac{x^2 - y^2}{x - a}$ (ii) $\lim_{x\to 1} \frac{1}{2^{x-1}}$ (iii) $\lim_{x\to 0} \frac{1}{1 - e^{1/x}}$

(iv) $\lim_{x\to 0} \frac{1}{1 - e^{\frac{1}{x-a}}}$ (v) $\lim_{x\to 2} \frac{x^2 - 3x + 2}{x - 2}$ (vi) $\lim_{x\to 0} \left(\frac{1}{x}\right)^{\frac{1}{e^x}}$

(vii) $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

2. If $f(x) = e^{\frac{1}{x}}$, then show that at $x = 0$, the right hand limits zero while the left hand limit is $+\infty$, and thus there is no limit of the function at $x = 0$.

3. Discuss the continuity, of the following function.

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases} \qquad \text{at } x = 1.$$

4. Discuss the continuity of $f(x)$ at $x = a$ where $f(x)$ is define as follows :

$$f(x) = \begin{cases} (x - a)\sin\dfrac{1}{x - a}, & x \neq a \\ 0, & x = a \end{cases}$$

5. Examine $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$ for continuity at x = 2.

6. Show that $|x|$ is continuous at $x = 0$ and draw its graph.

7. Investigate the continuity of the function :

$$f(x) = \begin{cases} \dfrac{x^2}{a} - a, & x < 2 \\ 0, & x = a \text{ at } x = a \\ a - \dfrac{a^2}{x}, & x < a \end{cases}$$

8. Examine $f(x) = \begin{cases} \dfrac{1}{e^{(x-2)^2}}, & x \neq 2 \\ 0, & x = 2 \end{cases}$ continuity at $x = 2$.

9. Examine whether or not the function

$$f(x) = \begin{cases} \dfrac{\sin 2x}{2}, & x \neq 0 \\ 1, & x = 0 \end{cases} \quad \text{is continuous at } x = 0$$

10. If f be a function defined on [0, 1] by

$$f(x) = \begin{cases} x, \text{ if } x \text{ is irrational} \\ 0, \text{ if } x \text{ is rational} \end{cases} \quad \text{then show that } f \text{ is continuous at } x = 0.$$

11. If f is a function defined on R as

$$f(x) = \begin{cases} \dfrac{[e^{\frac{1}{x}} - e^{\frac{1}{x}}]}{[e^{\frac{1}{x}} + e^{-\frac{1}{x}}]}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

then show that ! is discontinuous it $x = 0$.

12. Show that the following function is discontinuous at $x = 0$.

$$f(x) = \begin{cases} \dfrac{e^{\frac{1}{x}}}{1 + e^{-\frac{1}{x}}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

13. Define the continuity of a function at a point.Examine for continuity the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \text{at } x = 0$$

14. Discuss the continuity of the function

$$f(x) = \begin{cases} \frac{1}{2} - x, \text{ when } 0 < x < \frac{1}{2} \\ \frac{1}{2} \quad \text{when } x = \frac{1}{2} \text{ at } x = \frac{1}{2} \\ \frac{3}{2} - x \text{ when } \frac{1}{2} < x < 1 \end{cases}$$

15. A function g is defined by

$$f(x) = \begin{cases} \frac{1 - \cos x}{x^2}, \ x \neq 0 \\ k, \ x = 0 \end{cases}$$

Find the value of k if g is continuous at $x = 0$.

16. A function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} 3 + 2x, \text{ for } -\frac{3}{2} \leq x < 0 \\ 3 - 2x, \text{ for } 0 \leq x < \frac{3}{2} \\ -3 - 2x \text{ for } x \geq \frac{3}{2} \end{cases}$$

Show that $f(x)$ is continuous at $x = 0$ and is discontinuous at $x = \frac{3}{2}$

17. A function $f(x)$ is defined in the interval [0, 3] in the following way :

$$f(x) = \begin{cases} x^2, \text{ when } 0 < x < 1 \\ x, \text{ when } 1 \leq x < 2 \\ \frac{x^2}{4} \text{ when } 2 \leq x < 3 \end{cases}$$

Show that $f(x)$ is continuous at $x = 2$ and $x = 1$.

18. Prove that a continuous function on $[a, b]$ is always bounded, but a converse is not true.

ANSWERS

1. (i) $2a$ (ii) No, $f(1-0)$ 0, $f(1+0) = \infty$

(iii) No, $f(0+0) = 0$, and $f(a-0) = 1$

(iv) No, $f(a+0) = 0$, and $f(a-0) = 1$

(v) 1 (vi) No (vii) No

3. Continuous **4.** Continuous **5.** Continuous **7.** Continuous **8.** Continuous

9. Continuous **10.** No **13.** Continuous **14.** Discontinuous **15.** $k = \frac{1}{2}$

8.7. SUGGESTED READING

The students are advised to go through following references for details

8.8. REFERENCES

(1) Real analysis by by J.N. Kapur & H.C. Saxena, S.Chand & Co.

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8.9. MODEL TEST PAPER

Q.1. Do the following limits exist ? If they exist, find their values:

Q.2. Prove that every continuous function attains supremum & infimum

Q.3. A function g is defined by

$$f(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

Find the value of k if g is continuous at $x = 0$.

Q.4. A function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} \frac{1}{2} - x, & \text{when } 0 < x < \frac{1}{2} \\ 0 & \text{when } x = \frac{1}{2} \text{ at } x = \frac{1}{2} \\ \frac{3}{2} - 3x & \text{when } \frac{1}{2} < x < 1 \end{cases}$$

Show that $f(x)$ is continuous at $x = \frac{1}{2}$.

Q.5. A function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} 3 + 2x, & \text{for } -\frac{3}{2} \le x < 0 \\ 3 - 2x, & \text{for } 0 \le x < \frac{3}{2} \\ -3 - 2x & \text{for } x \ge \frac{3}{2} \end{cases}$$

Show that $f(x)$ is continuous at $x = 0$ and is discontinuous at $x = \frac{3}{2}$.

Q.6. A function $f(x)$ is defined in the interval [0, 3] in the following way :

$$f(x) = \begin{cases} x^2, & \text{when } 0 < x < 1 \\ x, & \text{when } 1 \le x < 2 \\ \frac{x^2}{4} & \text{when } 2 \le x < 3 \end{cases}$$

Show that f(x) is continuous at x = 2 and x = 1.

Q.7. Prove that a continuous function on $[a, b]$ is always bounded, but a converse is not true.

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DIFFERENTIABLE FUNCTIONS

9.1. Introduction : In this lesson the concept of differentiation of functions is discussed.

9.2. Objectives : Objective of studying this lesson is to explain differentiations of the functions & the difference between continuity & differentiation along with some of its properties.

9.3. DIFFERENTIABILITY AND MEAN VALUE THEOREMS DERIVATIVES OF A FUNCTION

Definition 9.3.1. If $f(x)$ is a finite and single valued function of x, then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if it exists, is called the derivative of $f(x)$ at $x = a$ and is denoted by $f'(a)$.

Equivalently, if $\lim_{h \to a} \frac{f(a+h) - f(a)}{h}$

exists, then it is denoted by $f'(a)$ and is called the derivative of $f(x)$ at $x = a$.

Right hand and left hand derivatives

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

means the same as

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if it exists is called the Right hand derivative at $x = a$, and is denoted by $f'(a + 0)$ or $Rf'(a)$.

Similarly $\lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$

means the same as $\lim_{h \to 0-0} \frac{f(a+h) - f(a)}{h}$

If it exists is called the Left hand derivative at $x = a$, and is denoted by $f'(a - 0)$ on Lf'(a).

If $Rf'(a)$ and $Lf'(a)$ both exist and are equal, then $f(x)$ is derivable at $x = a$ and the common value is nothing but $f'(a)$.

Remarks 9.3.2. (i) If $Rf'(a)$ and $Lf'(a)$ both exists and are different, then the derivative will not exist and the function will not be derivable at $x = a$.

(ii) If $f(x)$ possesses a derivative at every point of the interval (a, b), then it is said to a derivable in the interva1 (a, b).

(iii) If $f(x)$ is derivable on (a, b) and also at points a and b, then we say that $f(x)$ is derivable in $[a, b]$.

(iv) The process of finding the derivative of a function is called the Differentiability.

(v) Geometrically, the derivative of the function at a point represents the slope of the tangent at that point.

Example 9.3.3. Prove that $f(x) = x$ for all $x \in R$ is derivable in R, the set of real numbers.

Solution. If a is any point in R, then

$$f(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{a+h-a}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

Thus $f'(a) = 1$. Since a is any point of R, this means that $f'(x) = 1$ for all $x \in R$. Hence $f(x)$ is derivable for all $x \in R$.

Example 9.3.4. If n is any fixed positive integer and let f be the function defined on R by $f(x) = x^n$ for all $x \in R$, then f is derivable in R.

Solution. If a is any point of R, then

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{(a+h)'' - a''}{h}$$

$$= \lim_{h \to 0} \frac{a^n + {}^nc_1 a^{n-1}.h + {}^nc_2 a^{n-2}.h^2 + ... + h^n - a^n}{h}$$

$$= \lim_{h \to 0} \frac{h({}^nc_1 a^{n-1} + {}^nc_2 a^{n-2}.h + ... + h^{n-1})}{h}$$

$$= \lim_{h \to 0} (na^{n-1} + {}^nc_2 a^{n-2}.h + ... + h^{n-1})$$

$$= na^{n-1}$$

Thus $f'(a) = na^{n-1}$ for any $a \in R$

Hence $f'(x)$ exists for all $a \in R$

Example 9.3.5. Let $f(x) = |x|$. Then show that $f(x)$ is not derivable at $x = 0$.

Solution. By definition $|x| = \begin{cases} x \text{ when } x \geq 0 \\ -x \text{ when } x < 0 \end{cases}$

Here $f(0) = 0$

R$f'(0) = f'(0 + 0)$

$$= \lim_{x \to 0+0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+0} \frac{|x|}{x} = \lim_{x \to 0+0} \frac{x}{x} = 1$$

Then L$f'(0) = f'(0 - 0)$

$$= \lim_{x \to 0-0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-0} \frac{|x|}{x} = \lim_{x \to 0-0} \frac{-x}{x} = -1$$

Then L$f'(0) \neq$ R$f'(0)$

$\therefore$ f is not derivable at $x = 0$.

Example 9.3.6. Show that a function $f(x)$ defined as

$$f(x) = \begin{cases} x \quad \text{when } 0 \leq x < 1 \\ 2 - x \text{ when } x \geq 1 \end{cases}$$

is not differentiable at $x = 1$

Solution. Here $f(1)=1$

Now $\quad Lf'(1) = \lim_{h \to 0} \frac{(1-h)-1}{-h} = \lim_{h \to 0} \frac{-h}{-h} = 1$

$$Rf'(1) = \lim_{h \to 0} \frac{2-(1+h)-1}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$$

Then $Lf'(1) \neq Rf'(1)$

$\therefore$ The function is not differentiable at $x = 1$.

Theorem 9.3.7. If a function is derivable at a point, then it is continuous at that point.

Or

Differentiability $\Rightarrow$ Continuity.

Proof. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. Then for all $c \in [a, b]$.

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists and equal $f'(c)$. We shall show that $f(x)$ is continuous at $x = c$, For this consider

$$f(c+h) - f(c) = \frac{f(c+h) - f(c)}{h} \times h$$

$$\therefore \quad \lim_{h \to 0} [f(c+h) - f(c)] = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \times h$$

$$= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \times \lim_{h \to 0} h$$

$$= f'(c) \times 0 = 0$$

Thus $\lim_{h \to 0} [f(c+h) - f(c)] = 0 \quad \Rightarrow \quad \lim_{h \to 0} f(c+h) = f(c)$

This prove that $f(x)$ is continuous at $x = c$ for all $c \in [a, b]$.

Remark 9.3.8. The converse of the above theorem thus not hold, *i.e.* a function is continuous at a point but may fail to be derivable at that point.

In other words, continuity is a necessary condition for derivability but not sufficient as can be seen from the example given below.

9.4. EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

1. If c is any fixed number and f be the function defined on R by

   $f(x) = c$ for all $x \in$ R, then show that $f(x)$ is derivable for all $x \in$ R.

2. Show that the function $f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

   is continuous at $x = 0$, but is not differentiable at $x = 0$.

3. Show that the function

$$f(x) = \begin{cases} (x-a)\sin \dfrac{1}{x-a}, & \text{if } x \neq a \\ 0, & \text{if } x = a \end{cases}$$

   is continuous at $x = a$, but is not differentiable at $x = a$.

4. Show that the function

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

   is differentiable as well as continuous at $x = 0$.

5. Show that the function

$$f(x) = \begin{cases} 2 + x, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$ is not derivable at $x = 0$.

6. Show that the function f is defined on R as under :

$$f(x) = \begin{cases} 1, & \text{if } x \le 0 \\ 0, & \text{if } x < 0 \end{cases}$$ is not differentiable at $x = 0$.

7. Prove that every differentiable function is continuous. Is the converse true ?

8. Show that the function

$$f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$ is continous at $x = 0$ but is not derivable at $x = 0$.

9. Show that the function $f(x) = |x - 4|$ is continuous but not derivable at $x = 4$.

10. Examine the derivability of the function

$$f(x) = \begin{cases} -x^2 \text{ if } x \le 0 \\ 5\text{x} - 4 \text{ if } 0 < x \le 1 \\ 4x^2 - 3x \text{ if } 1 < x \le 2 \\ 3x + 4 \text{ if } x \ge 2 \end{cases}$$ at $x = 0$, 1 and 2.

9.5. MEAN VALUE THEOREMS

9.5.1.Rolle's Theorem. The following theorem, known as Rolle's theorem is one of the most important theorem of real analysis. It is at the root of all mean value theorems such as:

Taylor's theorem and Maclaurin's theorem which we shall discuss in the next lesson.

9.5.2. Rolle's Theorem. Let f be a function defined on [a, b] such that

(i) f is continuous on $[a, b]$ (ii) f is derivable on (a, b)

(iii) $f(a) = f(b)$

Then there exists a real number c between a and b such that $f'(c) = 0$.

Proof. Since f is continuous on $[a, b]$ and every continuous function on $[a, b]$ is bounded on $[a, b]$. Therefore f is bounded on $[a, b]$.

Let M = Sup. f, m = inf. f

Two different cases arise :

1. M = m. Then f is constant over $[a, b]$ and consequently

$f'(x) = 0$ for all $x \in [a, b]$.

2. M $\neq m$. Since $f(a) = f(b)$ therefore, at least one of the number M and m is differ from $f(a)$ and therefore, also from $f(b)$. For the sake of definiteness, assume that M $\neq f(a)$.

Since every continuous function on $[a, b]$ attains its supremum therefore, there exists some real number c in $[a, b]$ such that $f(c)$= M. Further, since $f(a) \neq$ M $\neq f(b)$, therefore, c is different from both a and b. This means that c lies in the open interval (a, b).

Since $f(c)$ is the supremum of f on $[a, b]$, therefore, $f(x) \leq f'(c)$ for all x in $[a, b]$. This means that

$$\frac{f(c-h)-f(c)}{-h} \geq 0 \qquad \text{...(i)}$$

For all positive real numbers h such that $c - h$ lies in $[a, b]$.

Taking limit as $h \to 0$ and observing that since $f'(x)$ exists at each point of (a, b), and therefore, in particular at $x = c$, we have

$$Lf'(c) \geq 0 \qquad \text{...(ii)}$$

From (i) we similarly have,

$$f(c + h) \leq f(c)$$

for all positive real numbers h such that $c + h$ lies in $[a, b]$. By the same argument as we have

$$Rf'(c) \leq 0 \qquad \text{...(iii)}$$

Since $f'(x)$ exists at $x = c$, therefore

$$Lf'(c) = f'(c) = Rf'(c) \qquad \text{...(iv)}$$

From (ii), (iii) and (iv) we find that $f'(c) = 0$.

Alternative Form of Rolle's Theorem

If a function $f(x)$ is such that

(i) it is continuous in the closed interval

(ii) it is derivable in the open interval

(iii) $f(a) = f(a + h)$

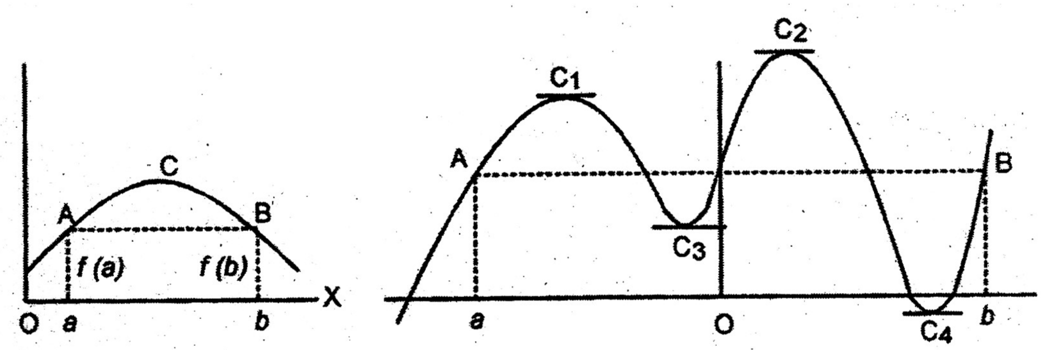
then there exists at least one number such that

$$f'(a + \theta h) = 0, \ 0 < \theta < 1$$

(Because the number c which lies between a and $a + h$ must be greater than a by a fraction of h and may be written as $c = a + h$ where $0 < \theta < 1$.

Note : Rolle's theorem fails to hold good for a function which does not satisfy even one three conditions stated above.

9.5.3. Geometrical Significance of Rolle's Theorem. When geometrically interpreted, the conclusion of the theorem states that the ordinates of the end point A, B being equal, there is a point on the curve the tangent at which is parallel to be cord AB (x-axis).



Example 9.5.4. Verify Rolle's Theorem for the function

$f(x) = x^2 - 6x + 8$ in the interval [2, 4].

Solution. Here $a = 2$, $b = 4$

1. $f(x) = x^2 - 6x + 8$ $f(x)$ is a polynomial. Since every polynomial is a continuous function of x for every value of x.

$f(x)$ is continuous in the closed interval [2, 4].

2. $f'(x) = 2x - 6$ which exists in the open interval (2, 4).

3. $f(2) = 4 - 12 + 8 = 0$

$f(4) = 16 - 24 + 8 = 0$

$f(2) = 0 = f(4)$

$f(x)$ satisfies all the conditions of Rolle's Theorem. Hence there must exist at least one number c between 2 and 4 such that $f'(c) = 0$.

Now $\quad f'(x) = 2x - 6$. Therefore $f'(c) = 0$ gives $2c - 6 = 0$, $c = 3$.

This is a point in the open interval (2, 4) and therefore, the theorem is verified.

Example 9.5.5. Discuss the applicability of Role's theorem of the function

$$f(x) = 2 + (x - 1)^{2/3} \text{ in } [0, 2].$$

Solution. Here $f(x) = 2 + (x - 1)^{2/3}$

$$f'(x) = \frac{2}{3}(x - 1)^{-1/3} \qquad \text{...(1)}$$

Equation (1) shows that $f'(x)$ does not exist at $x = 1 \in (0, 2)$. Therefore Rolle's theorem cannot be applied.

9.6. EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

1. Verify Rolle's Theorem for the function $(x - a)^3 (x - b)^4$ in the interval $[a, b]$.
2. Verify Rolle's Theorem for $x^3 - 4x$ for the interval $[-2, 2]$.
3. Verify Rolle's Theorem for the function $f(x) = 8x - x^2$ in $[0, 8]$.
4. Verify Rolle's Theorem for the function $f(x) = x(x + 3)e^{x/2}$ in $[-3, 0]$.
5. Verify Rolle's Theorem for the following functions :

   (i) $f(x) = \sin x$ in $[-\pi, \pi]$ (ii) $f(x) = e^x \sin x$ in $[0, \pi]$

   (iii) $f(x) = \log x$ $[0, e]$
6. Discuss the applicability of Rolle's Theorem to the function $f(x) = [x]$ in $[-1, 1]$.
7. Can Rolle's Theorem be applied to

   (i) $f(x) = \tan x$ in $[0, \pi]$ (ii) $f(x) = \sec x$ in $[0, 2\pi]$

ANSWER

1. $c = (3b + 4a)$ 2. $c = 1.555$ (approx.) 3. $c = 4$ 4. $C = -2$

5. (i) $c = \frac{\pi}{2}$ (ii) $c = \frac{3}{4}\pi$ (iii) $c = \frac{\pi}{4}$ 6. not applicable

7. (i) Rolle's theorem cannot be applied. (ii) Rolle's theorem cannot be applied.

9.7. LAGRANGE MEAN VALUE THEOREM

STATEMENT. If a function f (x) is such that

(i) it is continuous in the closed interval $[a, b]$

(ii) it is derivable in the open interval (a, b), then there exists at least one value c in open interval (a, b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

Proof. Consider the function

$$F(x) = f(x) + Ax \quad ...(i)$$

where A is the constant to be determined such that

$$F(a) = F(b)$$

Now $F(a) = f(a) + Aa$, $F(b) = f(b) + Ab$

Since $F(a) = F(b)$

$\therefore$ f(a)+Aa = f(b) +Ab

or $f(b) - f(a) = -A(b-a)$

$$-A = \frac{f(b)-f(a)}{b-a} \quad ...(ii)$$

Now $f(x)$ is given to be continuous in $a \le x \le b$ and derivable in $a < x < b$.

Also, A being constant, Ax is also continuous at $a \le x \le b$ and derivable in $a \le x \le b$.

$$F(x) = [f(x) + Ax] \text{ is}$$

1. Continuous in the interval $a \le x \le b$.

2. derivable in the interval $a < x \le b$.

3. F(a) = F(b)

$\therefore$ F satisfies all the three conditions of Rolle's Theorem.Thus there must exist one value c in the open interval (a, b) such that $F'(c) = 0$. Now $F'(x) = f'(x) + A$

$F'(c) = 0$ gives $f'(c) + A = 0$

or $\quad -A = f'(c) \quad ...(iii)$

From (ii) and (iii) we get

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Alternative form of Lagrange's Mean Value Theorem.

If a function $f(x)$ is such that

1. it is continuous in the closed interval $[a, a + h]$

2. it is derivable in the open interval $]a, a + h[$, then there exists at least one number θh that $f(a + h) = f(a) + hf'(a + \theta h)$ where $0 < \theta < 1$.

Proof. Let $a + h = b$.

Proved the first form $\frac{f(b)-f(a)}{b-a} = f'(c)$ ...(i)

Because $a + h = b$

$\therefore$ $b - a = h$, the length of the interval. The number c which lies between a and $a + h$ must be greater than 'a' by a fraction of h and may be written as c a + Oh where 0 is true positive fraction lying between 0 and 1, Let 0 < 0 <1.

(1) becomes $\frac{f(a+h)-f(a)}{b-a} = f'(a+\theta h)$

or $f(a+h) = f(a) \pm hf'(a+\theta h)$ where $0 < \theta < 1$.

9.7.1. Geometrical Interpretation of Lagrange's wean Value Theorem. Let A and B be points on the graph of the function $y = f(x)$ corresponding to $x = a$ and $x = b$. Therefore the coordinates of the points A and B are $[a, f(a)]$ and $[b, f(b)]$ respectively.

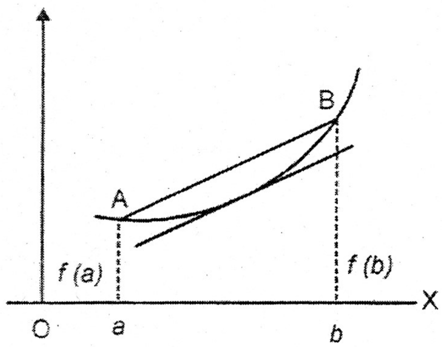
Slope of chord AB = $\frac{\text{difference of ordinates}}{\text{difference of abcissae}} = \frac{f(b)-f(a)}{b-a}$

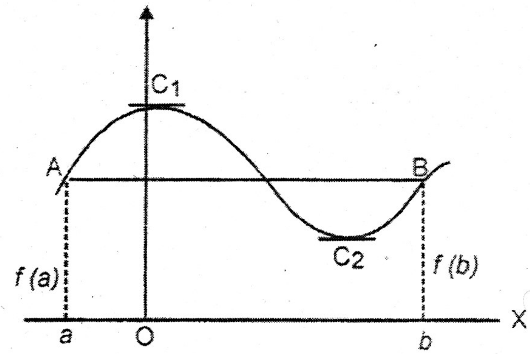
Also slope of the tangent at any point P, for which $x = c$, is $f'(c)$.

By Lagrange's mean value theorem, we have

.... $= f'(c)$, $a < c < b$

Slope of chord AB = slope of tangent at $x = c$





Thus Lagrange's Mean value theorem asserts geometrically that there exists at least point on the graph of the function at which the tangent is parallel to the chord joining the points A and B.

9.8. SUGGESTED READING

The students are advised to go through following references for details

9.8. REFERENCES

(1) Real analysis by by J.N. Kapur & H.C. Saxena, S.Chand & Co.

(2) Real analysis by J.N. Sharma & A.R.Vashishtha,Krishna Publication, New Delhi.

(3) Real analysis by Richard R. Goldberg, Oxford & IBH Publication Co. Pvt. Ltd. New Delhi.

(4) A text Book of Real Analysis by Sunil Gupta, Narinder Kumar, Malhotra Brothers Pacca Danga, Jammu.

(5) Real analysis by Jagdish Parsad Mittal & Neeraj Doda, Sharma Publication, Jallandhar.

9.9. MODEL TEST PAPER

Q.1. Verify Rolle's Theorem for the following functions :

(i) $f(x) = \sin x$ in $[-\pi, \pi]$ (ii) $f(x) = e^x \sin x$ in $[0, \pi]$

(iii) $f(x) =$ in $[0, \pi]$

Q.2. Discuss the applicability of Rolle's Theorem to the function $f(x) = [x]$ in $[-1, 1]$.

Q.3. Can Rolle's Theorem be applied to

(i) $f(x) = \tan x$ in $[0, \pi]$ (ii) $f(x) = \sec x$ in $[0, 2\pi]$

Prove that every differentiable function is continuous. Is the converse true?

Q.4. Show that the function is continous at origin but is not derivable at $x = 0$.

Q.5. Show that the function $f(x) = |x - 4|$

Is continuous butnot derivable at $x = 4$.

Q.6. Examine the derivability of the function ____________

$$f(x) = \begin{cases} -x^2 \text{ if } x \le 0 \\ 5x - 4 \text{ if } 0 < x \le 1 \\ 4x^2 - 3x \text{ if } 1 < x \le 2 \\ 3x + 4 \text{ if } x \ge 2 \end{cases} \quad \text{at } x = 0, 1 \text{ and } 2.$$

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APPLICATIONS OF DIFFERENTIABLE FUNCTIONS

10.1. Introduction: In this lesson some applications of mean value theorem are discussed.

10.2 Objectives : The objective of studying this lesson is to explain the expansions of some of important series of trigonometric functions.

10.3. CAUCHY'S MEAN VALUETHEOREM STATEMENT

If functions $f(x)$ and $g(x)$ such that

(i) both are continuous in the closed interval $[a, b]$.

(ii) both are derivable in the open interval (a, b).

(iii) $g'(x) \neq 0$ for any value of x in the open interval (a, b), then there exists at least one valuec of c in the open interval (a, b) such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f(c)}{g(c)}$.

Proof. Consider the function

$$F(x) = f'(x) + Ag(x) \qquad \text{...(i)}$$

where A is constant to be determined such that

$$F(a) = F(b)$$

Now $\quad F(a) = f(a) + Ag(a)$

$\quad F(b) = f(b) + Ag(b)$

Since $F(a) = F(b)$

$\therefore \quad f(a) + Ag(a) = f(b) + Ag(b)$

or $\quad f(b) - f(a) = -A[g(b) - g(a)]$

$\therefore \qquad -A = \dfrac{f(b) - f(a)}{g(b) - g(a)}$ ...(ii)

where $g(b) - g(a) \neq 0$, because if $g(b) - g(a) = 0$, then $g(a) = g(b)$. Therefore
$g(x)$ satisfies all the three conditions of Rolle's theorem $\Rightarrow g'(x) = 0$ for at least one value of x in the open interval $a < x < b$ which is contrary to the given condition that $g'(x) \neq 0$ for any value of x in the interval $a < x < b$.

Since $f(x)$ and $g(x)$ are both, given to the continuous in the interval and $a \leq 9x \leq b$, derivable in the interval a < x < b.

$\therefore$ $F(x) = f(x) + Ag(x)$ is

1. Continuous in the interval $a \leq x \leq b$.
2. derivable in the interval $a < x < b$.
3. $F(a) = F(b)$

$\therefore$ $F(x)$ satisfies all the three conditions of Rolle's Theorem.

Thus there exists at least one value c in the interval $a < x < b$ such that $F'(c) = 0$

Now $F'(x) = f'(x) + Ag'(x)$

$\therefore$ $F'(c) = 0$ gives $f'(c) + Ag'(c) = 0$.

$\therefore \qquad -A = \dfrac{f'(c)}{g'(c)}$ ...(iii)

From (ii) and (iii), we get

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Corollary 10.3.1.Derive Lagrange's Mean Value Theorem from Cauchy's Mean Value Theorem.

Proof : If $g(x) = x$, then $g(b) = b$, $g(a) = a$ and $g'(x) = 1$ for all x. Therefore the result of Cauchy's Mean Value Theorem viz.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

reduces to $\dfrac{f(b) - f(a)}{b - a} = \dfrac{f'(c)}{1} = f'(c)$ which is Lagrange's Mean Value Theorem.

Alternate form of Cauchy's Mean Value Theorem. If two functions *f (x)* and *g (x)* are such that

(i) both are continuous in the closed interval $[a, a + h]$

(ii) both are derivable in the open interval $(a, a + h)$.

(iii) $g'(x) \neq 0$ for any value of x in the open interval $(a, a + h)$, there exists at least one number such that

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)} \quad \text{where } 0 < \theta < 1.$$

Physical Interpretation. We may write

$$\frac{\{f(\mathrm{b}) - f(a)\} / (b-a)}{\{g(a+b) - g(a)\} / (b-a)} = \frac{f'(c)}{g'(c)}$$

Hence, the ratio of the mean rates of increase of two functions in an interval is equal to the ratio of the actual rates of increase of the functions at some point within the interval.

Example 10.3.2. Verify Lagrange's Mean Value theorem for the function

$$f(x) = x(x-1)(x-2) \text{ in } \left[0, \frac{1}{2}\right].$$

Solution. $f(x) = x(x-1)(x-2)$

$= x(x^2 - 3x + 2)$

$= x^3 - 3x^2 + 2x$

$a = 0,\ b = \frac{1}{2}$

1. $f(x)$ being a polynomial is continuous in the interval $0 \leq x \leq \frac{1}{2}$

2. $f'(x) = 3x^2 - 6x + 2$ which exists in the interval $0 < x < \frac{1}{2}$.

Therefore by Lagrange's Mean Value Theorem, we have

$$f' \ (c) \ = \ \frac{f\,(b) - f\,(a)}{b - a}$$

i.e. $$3c^2 - 6c + 2 = \frac{\left(\frac{1}{8} - \frac{3}{4} + 1\right) - 0}{\frac{1}{2} - 0} = \frac{3}{4}$$

or $$12c^2 - 24c + 5 = 0$$

$$c = \frac{24 \pm \sqrt{576 - 240}}{24} = \frac{24 \pm \sqrt{336}}{24} = \frac{24 \pm \sqrt[4]{21}}{24}$$

$$= 1 \pm \frac{1}{6}\sqrt{21} = 1 \pm \frac{1}{6}(4.58) = 1 \pm .76 = 1.76, \ .24$$

Discarding the value $c = 1.76$ which does not lie in the given interval $\left(0, \frac{1}{2}\right) = (0, .5)$.

$\therefore$ $c = 24$, a value which lies between 0 and $\frac{1}{2}$.

Hence the verification.

Example 10.3.3. Find c of Cauchy's Mean Value Theorem for the pair of functions

$$f\,(x) = \sqrt{x}, \ g\,(x) = \frac{1}{\sqrt{x}} \text{ in} [a, \ b].$$

Solution. $f\,(x) = x, \ g\,(x) = \frac{1}{\sqrt{x}}$ [Assuming $0 < a < b$].

$$f'\,(x) = \frac{1}{2\sqrt{x}} g'\,(x) = -\frac{1}{2x\sqrt{x}}$$

Both $f\,(x)$ and $g\,(x)$ are continuous in $[a, \ b]$ and derivable in $(a, \ b)$.

By Cauchy's Mean Value Theorem we have

$$\frac{f\,(b) - f\,(a)}{g\,(b) - g\,(a)} = \frac{f'\,(c)}{g'\,(c)}$$

or
$$\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}=\frac{\frac{1}{2\sqrt{c}}}{\frac{1}{2c\sqrt{c}}}$$

or
$$\sqrt{b}-\sqrt{a}\,\frac{\sqrt{ab}}{\sqrt{a}-\sqrt{b}}=-c$$

or
$$-\sqrt{ab}=-c$$

$$c=\sqrt{ab}\in(a, b)$$

10.3.4. Important Deduction from the Mean Value Theorem.

1. If $f(x)$ be a function such that $f'(x)=0$ for all values of x in $a < x < b$, then $f(x)$ is a constant in this interval.

Proof. Let x_1, x_2 be any two values of x such that $a < x_1 < x_2 < b$.

Because $f'(x) = 0$ for all values of x in (a, b) and $\{x_1, x_2\} \subseteq (a, b)$. Since $f(x)$ satisfies all the condition of the Lagrange's Mean Value Theorem in $[x, x_2]$ therefore we have

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c) \text{ where } x_1 < c < x_2 .$$

But $f'(x) = 0$ for all x in (a, b). Therefore $f'(c) = 0$

$\Rightarrow \quad f(x_2)-f(x_1)=(x_2-x_1)\times 0=0 \quad$ *i.e.* $\quad f(x_2)=f(x_1)$

Since x_1 and x_2 are any two values of x in (a, b), it follows that $f(x)$ has the same value for every value of x in (a, b). Hence $f(x)$ is a constant in the interval (a, b).

Corollary 10.3.5. If two functions $f(x)$ and $g(x)$ have the same derivatives. Then they differ by a constant.

Proof : Consider a function

$$F(x) = f(x) - g(x) \text{ where } f'(x) = g'(x)$$

Now $\quad F'(x) = f'(x) - g'(x) = 0$

By Deduction I, $F(x) = c$, a constant *i.e.*, $f(x) - g(x) = c$.

2. If the derivative $f'(x)$ is positive or zero in (a, b), without being always zero, then

$f(b) > f(a)$.

Proof : Let x_1, x_2 be any value between a and b;then applying Mean Value Theorem to the function $f(x)$ for the two intervals $[a, x]$ and $[x, b]$, we get

$$\frac{f(x) - f(a)}{x - a} = f'(c_1) \text{ and } \frac{f(b) - f(x)}{b - x} = f'(c_2)$$

where $a < c_1 < x$ and $x < c_2 < b$.

But $f'(c_1) \geq 0$ and $f'(c_2) \geq 0$

Therefore, we get

$$f(x) - f(a) \geq 0 \text{ and } f(b) - f(x) \geq 0$$

$$\Rightarrow \quad f(x) \geq f(a) \text{ and } f(b) \geq f(x)$$

$$\Rightarrow \quad f(b) \geq f(x) \geq f(a)$$

$$\Rightarrow \quad f(b) \geq f(a)$$

But $f(b) \neq f(a)$

For, if it were so, then $f(x) = f(b) \ \forall x \in [a, b]$ and the function reduces to a constant whose derivative is always equal to zero, which contrary to the hypothesis that $f'(x)$ is not zero in $[a, b]$. Hence $f(a) > f(b)$.

3. If the derivative $f'(x)$ is negative or zero in $[a, b]$, without being zero always, then $f(b) < f(a)$. The proof is similar to (2).

Note. Increasing or decreasing function. A function $f(x)$ in the interval (a, b) is said increasing or decreasing function according as

$$f(x_2) > f(x_1) \quad \text{or} \quad f(x_2) < f(x_1) \quad \text{where} \quad a \leq x_1 < x_2 \leq b.$$

10.4 EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

Q. Verify Lagrange's Mean Value Theorem for the following functions and find c if possible

1. $f(x) = (x - 1)(x - 2)(x - 3)$ in $[0, 4]$.

2. $f(x) = \sqrt{x^2 - 4}$ in $[2, 4]$.

3. $f(x) = \log x$ in $[1, e]$

4. $f(x) = e^x$ in $[0, 1]$.

5. $f(x) = x^3 - 5x^2 - 3x$ in [1, 3].

6. $f(x) = \dfrac{2x-1}{3x-4}$ in [1,2].

Find 'c' of Cauchy's Mean, Value Theorem for the following pairs of function in [a, b].

7. $f(x) = e^x$, $g(x) = e^{-x}$

8. $f(x)$ sin x, $g(x) = \cos x$

9. Verify Cauchy's Mean Value Theorem for the functions $f(x) = x^2$ and $g(x) = x^3$ in [1, 2].

10. If in Cauchy's Mean Value Theorem we write

$f(x) = \dfrac{1}{x^2}$, $g(x) = \dfrac{1}{x}$ then show that c is the harmonic mean between a and b.

ANSWERS

1. $c = 3.155, .845$ 2. $c = \sqrt{6}$ 3. $c = e - 1$

4. $c = \log(e - 1)$ 5. $c = \dfrac{7}{3}$

6. Theorem fails as there is no value of c in (1, 2) that satisfies the conditions of Theorem.

7. $c = \dfrac{a+b}{2}$ 8. $c = \dfrac{a+b}{2}$ 9. $c = \dfrac{14}{9}$

10.5. TAYLOR'S THEOREM WITH LAGRANGE'S FORM OF REMAINDER AFTER N–TERMS

Statement. If a function $f(x)$ is such that

1. $f(x), f'(x), f''(x), \ldots\ldots\ldots, f^{n-1}(x)$ are continuous in the closed interval $a \le x \le a + h$.

2. $f^n(x)$ exists in the open interval $a < x < a + h$,

then there exists at least one number 0 between 0 and 1 such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} - f''(a) + \frac{h^3}{3!} - f'''(a) ..... + \frac{h^{n-1}}{(n-1)} f^{n-1}(a) + \frac{h^n}{n!} - f^n(a+\theta h)$$

Proof. Consider the function

$$F(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{2!} f''(x) + ...$$

$$+ \frac{(a+h-x)^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{(a+h-x)^n}{(n)!} . A \qquad ...(i)$$

where A is a constant to be determined such that

F (a) = F $(a + h)$

Now F $(a) = f(a) + hf'(a) + ...\ \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{(n)!} A$

F $(a + h) = f(a + h)$

Since F $(a + h)$ = F (a)

$$\therefore \quad f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} A \qquad ...(ii)$$

Now it is given that $f(x), f'(x), f''(x), ..., f^{n-1}(x),$ are continuous in the interval $a \le x \le a+h$ and their derivatives $f'(x), f''(x), ..., f^n(x)$.

Also $(a+h-x), \frac{(a+h-x)^2}{2!}, ..., \frac{(a+h-x)^n}{n!}$

(being polynomials) and A (being constant)

are continuous in the interval $a \le x \le a+h$ and derivable in the interval $a < x < a + h$.

$\therefore$ F (x) is 1. Continuous in the closed interval $a \le x \le a+h$

2. derivable in the open interval $a < x < a + h$,

and 3. F $(a + h)$ = F (a).

Thus F (x) satisfies all the three conditions of Rolle's Theorem. Therefore there exists at one number θ between 0 and 1 such that

$F'(a + \theta h) = 0.$

Now $F'(x) = f'(x) - f'(x) + (a+h-x) f''(x) - (a+h-x) f''(x) + \ldots\ldots\ldots\ldots$

$$\ldots\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!} f^n(x) - \frac{(a+h-x)^{n-1}}{(n-1)!} A$$

$$= \frac{(a+h-x)^{n-1}}{(n-1)!}[f^n(x) - A]$$

But $F'(a+\theta h) = 0$ gives $\dfrac{(a+h-a-\theta h)^{n-1}}{(n-1)!}[f^n(x) - A]$

But $F'(a + h) = 0$ gives $\dfrac{(a+h-a-\theta h)^{n-1}}{(n-1)!}[F^n(a+\theta h) - A] = 0$

Now $h \neq 0$ and $1 - \theta \neq 0$ $[\because \; 0 < \theta < 1]$

$\therefore \quad A = f^n(a + \theta h)$

From (ii), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

Then (n + 1)th term $\dfrac{h^n}{n!} f^n(a+\theta h)$ is called Lagrange's form of remainder after n-terms and is denoted by R_n.

10.6. MACLAURIN'S THEOREM WITH LAGRANGE'S FORM OF REMAINDER AFTER *n*–TERMS

Statement. If a function $f(x)$ is such that

1. $f(x), f'(x), f''(x), \ldots\ldots, f^{n-1}(x)$ are continuous in the closed interval [0, x].

2. $f^n(x)$ exists in the open interval (0, x) then there exists at least one number

between 0 and 1 such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(a) + ... + \frac{x^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{x^n}{(n)!} f^n(a + \theta x)$$

This we can get by putting $a = 0$ and $h = x$ in Taylor's Theorem.

Taylor's and Maclaurin's Series

10.6.1. Taylor Series. Let the function $f(x)$ possesses derivatives of all orders in an interval $[a, a + h]$, then for all positive integral values of n, we know that

$$f(a + h) = f(a) \pm xf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

where $R_n = \frac{h^n}{n!} f^n(a + \theta h),\ 0 < \theta < 1$

If now $R_n \to 0$ as $n \to \infty$, then

$$f(a + h) = \lim_{n \to \infty} f(a) + hf(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + R_n$$

where $R_n \to 0$ as $n \to \infty$, then

$$f(a + h) = \lim_{n \to \infty} \left[f(a) + hf(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) \right]$$

so that we see that the series

$$f(a) = hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$

is convergent and is sum is $f(a + h)$.

Thus we have shown that if $f(x)$ possesses derivatives of all orders in the interval $[a, a + h]$ and the remainder R_n, tends to zero as n-tends to infinitely, then

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + ... + \frac{x^{n-1}}{n!} f^n(0) + ...\qquad ...(A)$$

This series is called **Taylor's series.**

10.6.2. Maclaurin's Series. Put $a = 0$ and $h = x$ in (A), we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \text{.........} + \frac{x^n}{n!} f^n(0) + \text{.......} \qquad \text{..(B)}$$

This series is called **Maclaurin's series.**

Note. Put $h = x - a$ in (A), we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \text{.........} = \frac{(x-a)^n}{n!} f^n(a) + \text{.......} \qquad \text{...(C)}$$

This is another form of Taylor's series.

Example 10.6.3. Expand a^x by Maclaurin's theorem with Lagrange's form of remainder n-terms.

Solution. Here $f(x) = a^x \Rightarrow f'(x) = a^x \log a,\ f''(x) = a^x (\log a)^2$

$\therefore \quad f^n(x) = a^x (\log a)^n$

Putting $x = 0$, we get $f^n(0) = (\log a)^n$

$\therefore \quad f(0) = 1,\ f'(0) = \log a,\ f''(0) = (\log a)^2 \text{.........}$

$f^{n-1}(0) = (\log a)^{n-1}$ and $f^n(\theta x) = (a\theta)^x (\log a)^n$

By Maclaurin's Theorem with Lagrange's Form of remainder after n-terms, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(a) + \text{.........} + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

$(0 < \theta < 1)$

$$\therefore \quad a^x = 1 + x \log a + \frac{x^2}{2!} (\log a)^2 + \text{........} + \frac{x^{n-1}}{(n-1)!} (\log a)^{n-1} + \frac{x^n}{n!} (\log a)^n$$

Here Lagrange's remainder after n-terms $= \dfrac{x^n}{n!} (\log a)^n$ where $0 < \theta < 1$.

Example 10.6.5. Expand $\tan^{-1} x$ in powers of $\left(x - \frac{x}{4}\right)$.

Solution. By Taylor's series,we know that

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots\ldots \qquad \ldots(i)$$

Here $f(x) = \tan^{-1} x$ and $a = \frac{\pi}{4}$

$$f(a) = \tan^{-1} \frac{\pi}{4}$$

$f'(x) = \dfrac{1}{1+x^2}$	$f'(a) = \dfrac{1}{1+\left(\frac{\pi}{4}\right)^2}$
$f''(a) = \dfrac{-2x}{(1+x^2)^2}$	$f''(a) = \dfrac{-\frac{\pi}{2}}{\left(1+\frac{\pi^2}{16}\right)^2}$

Putting in (1), we get

$$\tan^{-1} x = \tan^{-1} \frac{\pi}{4} + \left(x - \frac{\pi}{4}\right) \frac{1}{1+\frac{\pi^2}{16}} + \frac{\left(x-\frac{\pi}{4}\right)}{2!} \frac{-\frac{\pi}{23}}{\left(1+\frac{\pi^2}{16}\right)^2} + \ldots\ldots\ldots$$

$$= \tan^{-1} \frac{\pi}{4} + \frac{1}{1+\frac{\pi^2}{16}}\left(x - \frac{\pi}{4}\right) - \frac{\pi}{4\left(1+\frac{\pi^2}{16}\right)}\left(x - \frac{\pi}{4}\right)^2 + \ldots\ldots\ldots$$

Example 10.6.6. Prove that

$$\sin^{-1}(x+h) = \sin^{-1} x + \frac{h}{\sqrt{1-x^2}} + \frac{x}{(1-x^2)^{3/2}}.\frac{h^2}{2!} + \frac{1+2x^2}{(1-x^2)^{5/2}}.\frac{h^3}{3!} + ...$$

Solution. Here $f(x + h) = \sin^{-1}(x + h)$

$$f(x) = \sin^{-1} x$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{2}(1-x^2)^{-\frac{3}{2}}.(-2x) = \frac{x}{(1-x^2)^{\frac{3}{2}}}$$

$$f'''(x) = \frac{1.(1-x^2)^{\frac{3}{2}} - x.\frac{3}{2}(1-x^2)^{\frac{1}{2}}(-2x)}{(1-x^2)^3} = \frac{(1-x^2)^{\frac{3}{2}} - 3.x^2(1-x^2)^{\frac{1}{2}}}{(1-x^2)^3}$$

$$= \frac{(1-x^2)^{\frac{1}{2}}[1-x^2+3x^2]}{(1-x^2)^3} = \frac{1+2x^2}{(1-x^2)^{5/2}}$$

By Taylor's series, we know that

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + ...$$

$$\sin^{-1}(x+h) = \sin^{-1} x + \frac{h}{\sqrt{1-x^2}} + \frac{h^2}{2!}.\frac{x}{(1-x^2)^{3/2}} + \frac{h^3}{3!}.\frac{1+2x^2}{(1-x^2)^{5/2}}$$

$$= \sin^{-2} x + \frac{h}{\sqrt{1-x^2}} + \frac{x}{(1-x^2)^{3/2}}\frac{h^2}{2!} + \frac{1+2x^2}{(1-x^2)^{5/2}}\frac{h^3}{3!} + ...$$

10.7. EXAMINATION ORIENTED EXERCISE/ LESSON END EXERCISE

Q.1. Expand e^x by Maclaurin's theorem with Lagrange'sform of remainder after n-terms.

Q.2. Show that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots + (-1)^{n-2}\frac{x^{n-1}}{n-1} + (-1)^{n-1}\frac{x^n}{n(1+\theta x)^n} \text{ for } x > -1$$

Q.3. Prove that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^3}{4!} + \ldots\ldots\ldots + \frac{(-1)^n x^{2n}}{(2n)!} + \ldots\ldots$ for all $x \in R$

Q.4. Find the Taylor's series about $x = 2$ for $f(x) = x^3 + 2x + 1$ $(-\infty < x < \infty)$.

Q.5. Expand (i) x^3 in powers of $(x - 1)$

(ii) $\sin x$ in powers of $(x - 4)$

(iii) x^n in powers of $(x - a)$.

Q.6. Assuming the possibility of expansions :

Prove the following :

(i) $e^{x+h} = e^x\left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots\ldots\ldots\right]$

(ii) $\tan^{-1}(x+h) = \tan^{-1}x + \frac{h}{1+x^2} - \frac{xh^2}{(1+x^2)^2} + \ldots\ldots.$

(iii) $\log\sin(x+h) = \log\sin x + h\cot x - \frac{1}{2}h^2\operatorname{cosec}^2 x + \frac{1}{2}h^3\cot x\operatorname{cosec}^2 x + \ldots..$

Q.7. By Maclaurin's theorem or otherwise, find the expansion of

$\sin(e^x - 1)$ upto and including the term in x^4.

Q.8. Assuming the possibility of expansion, obtain the following :

(i) $\log(1-x) = x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...$

(ii) $\sin ax = ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} - ....$

(iii) $\log\sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{46} + ...$

10.8. SUGGESTED READING

The students are advised to go through following references for details

10.9. REFERENCES

(1) Real analysis by by J.N. Kapur & H.C. Saxena, S.Chand & Co.

(2) Real analysis by J.N. Sharma & A.R.Vashishtha,Krishna Publication, New Delhi.

(3) Real analysis by Richard R. Goldberg, Oxford & IBH Publication Co. Pvt. Ltd. New Delhi.

(4) A text Book of Real Analysis by Sunil Gupta, Narinder Kumar, Malhotra Brothers Pacca Danga, Jammu.

(5) Real analysis by Jagdish Parsad Mittal & Neeraj Doda, Sharma Publication, Jallandhar.

10.10. MODEL TEST PAPER

Q.1. Expand e^x by Maclaurin's theorem with Lagrange'sform of remainder after n-terms.

Q.2. Show that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + ... + (-1)^{n-2}\frac{x^{n-1}}{n-1} + (-1)^{n-1}\frac{x^n}{n(1+\theta x)^n} \text{ for } x > -1$$

Q.3. Prove that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^3}{4!} + ... + \frac{(-1)^n x^{2n}}{(2n)!} + ...$ for all $x \in \mathrm{R}$

Q.4. Find the Taylor's series about $x = 2$ for $f(x) = x^3 + 2x + 1$ $(-\infty < x < \infty)$.

Q.5. Expand (i) x^3 in powers of $(x - 1)$

(ii) $\sin x$ in powers of $(x - 4)$

(iii) x^n in powers of $(x - a)$.

Q.6. Assuming the possibility of expansions :

Prove the following :

(i) $e^{x+h} = e^x\left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + ...\right]$

(ii) $\tan^{-1}(x + h) = \tan^{-1} x + \frac{h}{1 + x^2} - \frac{xh^2}{(1 + x^2)^2} + ...$

(iii) $\log\sin(x + h) = \log\sin x + h\cot x - \frac{1}{2}h^2\text{cosec}^2 x + \frac{1}{2}h^3\cot x\,\text{cosec}^2 x + ...$

Q.7. Assuming the possibility of expansion, obtain the following :

(i) $\log(1 - x) = x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...$

(ii) $\sin ax = ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} - ....$

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COMPLEX TRIGONOMETRY

11.1. Introduction : In this lesson the concept of De Moivre's theorem and its application is discussed.

11.2 Objectives : Objective of studying this lesson is to explain De Moivre's theorem and its application in solving problems.

11.3. COMPLEX NUMBERS

The students is already familiar with the idea of a complex numbers. In the domain of real numbers there is no number which satisfies the equation $x^2 = -1$. In order to enlarge our conception of number in such a way that it may be possible to apply the algebraical operation of root extraction to any number whatsoever, a new kind of number, denoted by i and known as the *imaginary unit* is introduced.

This number is defined as satisfying the fundamental laws of algebra, associative, commutative and distributive, and as being such that $i^2 = -1$.

This generalisation of the idea of number is valid one since no deductions from it lead to contradictions.

A number of the form $z = x + iy,$ where x and y are real numbers, is called a *complex number ;* x is called its *real part* and is denoted by R(z) , while y is called its *imaginary part* and is written as I (z).

If $y = 0$, the number is purely real; if $x = 0$, it is purely imaginary.

All the operations of algebra-addition, subtraction, multiplication, division, and root extraction. – apply to complex numbers, and they satisfy the fundamental laws, associative, commutative and distributive of these operations.

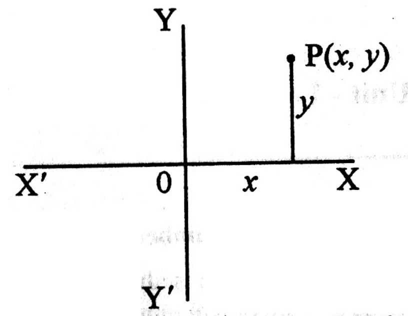
We also know that if two complex numbers are equal, then their real parts are equal and their imaginary parts are equal. In particular, the complex number $x + iy$ cannot have the value zero unless x *andy* are both zero.

11.3.1. Geometrical Representation of a Complex Number – The Argand Diagram.

We know how a real number can be represented by a point on a straight line.

With the complex number $z = x + iy$ are attached two real numbers x and y, occuring in a particular order *i.e.,* x coming first and y after it. In other words, with the complex number $z = x + iy$ is associated an *ordered pair (x, y) of real numbers.* Thus ordered pair of real numbers gives us a definite point in a plane with x as its abcissa and y as its ordinate. In this way we get a method of representing a complex number geometrically by a point in a plane.

Thus, the complex number $z = x + iy$ is represented geometrically by the point P whose rectangular co-ordinates are x and y.



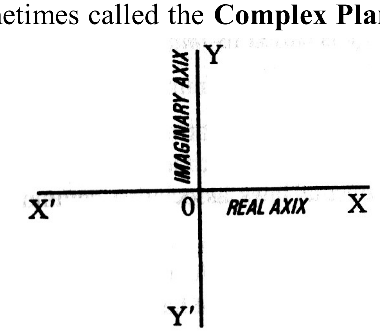
It is clear that the complex number $z = x + iy$ defines unique point P (x , y) and conversely the point P (x , y) defines a unique complex number $z = x + iy$.

The point P is said to be "the point corresponding to the complex number z" or simply "the point z".

This sort of geometrical representation of complex number by points in a plane was suggested by Argand, a Swiss Mathematician, in 1806, and so the diagram representing complex numbers by points is called the **Argand Diagram.**

The plane in which we draw this diagram is sometimes called the **Complex Plane.**

If the complex number $z = x + iy$ has its imaginary part $y = 0$, then it becomes purely real. In this case it is represented by the point $(s, 0)$ which lies in x-axis. Thus, purely real numbers are represented by points lying on x-axis. For the reason in an Argand diagram the x-axis is called the **real axis.**



Similarly, purely imaginary numbers are represented by points which lie on y-axis and, for this reason, in an Argand diagram the y-axis is called the **imaginary axis.**

Example 1. Find the points corresponding to the complex numbers

$$3 + 4l, \; -2 + 5i, \; -2 - 3i, \; 2 - 7i, \; 5, \; 6i.$$

Solution. The points are (3, 4), (– 2, 5), (– 2, – 3), (2, – 7), (5, 0) and (0, 6) respectively.

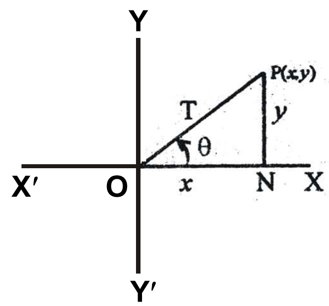
Example 2. Find the complex numbers corresponding to the points

(– 1, – 1), (0, – 2) and (– 3, 0).

Solution. The complex numbers are $-1-i$, $-2i$ and -3 respectively.

11.3.2. The Modulus and the Amplitude of a complex number. Let P be the point $z = x + iy$. Let the polar co-ordinates of P be (r, θ), where r is the positive measure on the length of OP, and θ is the measure of the angle XOP.

Then, $\dfrac{OM}{OP} = \cos\theta \Rightarrow \dfrac{x}{r} = \cos\theta \Rightarrow x = r\cos\theta$



Parallelly $\dfrac{PM}{OP} = \sin\theta$

$PM = OP \sin\theta$

$y = r\sin\theta$

$$\left.\begin{aligned} x &= r\cos\theta \\ \text{and} \quad y &= r\cos\theta \end{aligned}\right\} \qquad \ldots(1)$$

From these equations we get

$$\left.\begin{aligned} r &= \sqrt{x^2+y^2} \\ \text{and} \quad \theta &= \tan^{-1}\frac{y}{x} \end{aligned}\right\} \qquad \ldots(2)$$

The number r is called the **modulus** of z and is written as **mod** z or $|z|$, and the **number** θ is called the **amplitude** of z and written as **amp** z.

Thus, if $z = x + iy$, then $|z| = \sqrt{x^2+y^2}$ and amp $z = \tan^{-1}\dfrac{y}{x}$.

Cor. $|-z| = |-x-iy| = \sqrt{(-x)^2+(-y)^2} = \sqrt{x^2+y^2} = |z|$.

Principal value of the amplitude

Note 1. Obviously, θ = amp z has many values differing from one another by multiplies of 2θ.

The value of θ which lies between – π and π is called **principal value of the amplitude.**

As a rule, when we speak of the amplitude, we always mean the principal value of the amplitude.

Thus, $-\pi < \text{amp } z \le \pi$.

Note 2. The basic equations connecting (x, y) and (r, θ) are

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad \text{...(A)}$$

From these, by division, we obtain

$$\tan\theta = \frac{y}{x} \qquad \text{or} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad \text{...(B)}$$

The value of θ which satisfies the two equation of (A) simultaneously will satisfy (B), but all the values of θ which satisfy (B) may not satisfy (A).

Hence, θ should be obtained from (A) and not from (B).

Note 3. $z = r(\cos\theta + i\sin\theta)$ expresses the complex number z in terms of its modulus and amplitude.

It is called the **trignometric form of *z*.**

11.3.3. Example. Express the following complex numbers in trignometric form, indicating the modulus and amplitude in each case :

$$1 + i\sqrt{3},\ -1 + i\sqrt{3},\ -i\sqrt{3},\ -1 - i\sqrt{3},\ 2,\ -2,\ 2i,\ -2i$$

Sol. (*i*) $1 + i\sqrt{3}$

Here $\left.\begin{aligned} r\cos\theta &= 1 \\ \text{and} \quad r\sin\theta &= \sqrt{3} \end{aligned}\right\}$ ...(1)

Squaring and adding, we get $r^2 = 4$ or $r = 2$.

Substituting for *r*, we get

$$\left.\begin{aligned} \cos\theta &= \frac{1}{2} \\ \text{and} \quad \sin\theta &= \frac{\sqrt{3}}{2} \end{aligned}\right\}$$

$\therefore \qquad \theta = \frac{\pi}{3}$ (principal value)

Hence, $\qquad 1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

It may be noted that the modulus of the given complex number is 2 and $\frac{\pi}{3}$.

11.3.4. Modulus of a sum

Theorem 1. $|z_1 + z_2| \le |z_1| + |z_2|$

Proof. $|z_1 + z_2| \le |z_1| + |z_2|$

or if $\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \le \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$

or if

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 \le (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{x_1^2 + y_1^2}.\sqrt{x_2^2 + y_2^2}$$

or if $\qquad x_1x_2 + y_1y_2 \le \sqrt{x_1^2 + y_1^2} \,.\, \sqrt{x_2^2 + y_2^2}$

or if $\qquad (x_1x_2 + y_1y_2)^2 \le (x_1^2 + y_1^2)(x_2^2 + y_2^2)$

or if $\qquad 2x_1x_2y_1y_2 \le x_1^2y_2^2 + x_2^2y_1^2$

or if $\qquad 0 \le (x_1y_2 - x_2y_1)^2$

or if $\qquad (x_1y_2 - x_2y_1)^2 \ge 0$ which is always true.

Hence, the result.

Second Proof.

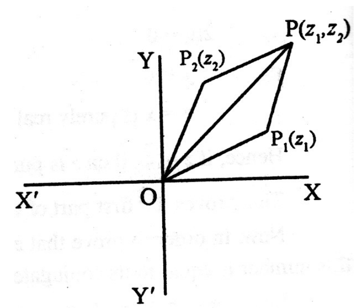
See the construction for the sum of two complex numbers

$$|z_1| = OP_1, \; |z_2| = OP_2 = P_1P,$$

and $\qquad |z_1 + z_2| = OP.$

In triangle OP_1P, we have

$$OP \leq OP_1 + P_1P$$



$\therefore \quad |z_1 + z_2| \leq |z_1| + |z_2|$

Cor. 1. The result can be extended step by step.

Thus, $\quad |z_1 + z_2 + z_3| \leq |z_1 + z_2| + |z_3|$

$$\leq |z_1| + |z_2| + |z_3|, \text{ etc.}$$

In general,

$$|z_1 + z_2 + \dots\dots + z_n| \leq |z_1| + |z_2| + \dots\dots + |z_n|$$

Cor 2. $\quad |z_1 - z_2| = |z_1 + (-z_2)| \leq |z_1| + |-z_2|$

But $\quad |-z_2| = |z_2|$

$\therefore \quad |z_1 - z_2| \leq |z_1| + |z_2|$

Cor 3. $\quad |z_1 - z_2| \geq |z_1| - |z_2|$

Proof. $\quad |z_1| = |(z_1 - z_2) + z_2| \leq |z_1 - z_2| + |z_2|$

or $\quad |z_1| - |z_2| \leq |z_1 - z_2|$

$\therefore \quad |z_1 - z_2| \geq |z_1| - |z_2|.$

11.3.5. Modulus and amplitude of a product

Theorem 3. $\quad |z_1 z_2| = |z_1| . |z_2|$

and $\quad \text{amp}(z_1 z_2) = \text{amp}(z_1) + \text{amp}(z_2).$

Proof. Let $\quad z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $\quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then, $\quad z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

$\therefore \quad |z_1 z_2| = r_1 r_2 = |z_1| . |z_2|$

Also, $\quad \text{amp}(z_1 z_2) = \theta_1 + \theta_2 = \text{amp}\, z_1 + \text{amp}\, z_2$

For any complex number $z = x + iy$

we have $\bar{z} = \overline{x + iy} = x - iy$ and $z + \bar{z} = 2x \Rightarrow 2\text{Re}\, z = z + \bar{z}$

Also $z - \bar{z} = 2iy \Rightarrow I_m z = \dfrac{z - \bar{z}}{2i}$

Also $z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 \geq 0$ always. The non-negative square root of $z\bar{z}$ is called **modulus** or the *absolute value* of the complex number z and denoted by

$$|z| = +\sqrt{z\bar{z}} \Rightarrow |z|^2 = z\bar{z}$$

Note that $|z| = |\bar{z}|$ and $R(z) \leq |z|$

11.4. DE-MOIVRE'S THEOREM

Statement of De Moivre's Theorem.

If θ is real and n is rational, then the value, or one of the values of

$$(\cos\theta + i\sin\theta)^n \text{ is } \cos n\theta + i\sin n\theta.$$

Proof : Case I. When *n* is positive integer.

By actual multiplication, we have

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$= \cos\theta_1 \cos\theta_1 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)$

$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$ ...(1)

Again, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$= [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)](\cos\theta_3 + i\sin\theta_3)$

$= \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$

Hence, by repeated multiplication and using of (1), we get

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots\ldots\ldots(\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \ldots\ldots\ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots\ldots\ldots + \theta_n)$

Now put $\theta_1 = \theta_2 = \ldots\ldots\ldots \theta_n = \theta$

We get, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Thus, if n is a positive integer, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Case II. When n is a negative integer.

Let $n = -m$, where m will be a positive integer.

Then, $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \quad \text{by case I}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos(-m\theta) + i\sin(-m\theta)$$

$$= \cos n\theta + i\sin n\theta$$

Thus, if n is a negative integer.

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta).$$

Case III. When n is a fraction, positive or negative.

In this case, we show that one of the values of

$$(\cos\theta + \mathrm{i}\sin\theta)^n \text{ is } \cos n\theta + i\sin n\theta.$$

Let $n = \frac{p}{q}$, where q is a positive integer, and p an integer positive or negative.

Suppose further that the fraction $\frac{p}{q}$ is in its lowest terms *i.e.,* p and q have no common factor.

Now $\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q = \cos\left(q\,.\,\frac{p}{q}\,\theta\right) + i\sin\left(q\,.\,\frac{p}{q}\theta\right)$, by case I

$$= \cos p\theta + i \sin p\theta$$

$$= (\cos\theta + i \sin\theta)^p \quad \text{by case I \& II}$$

$\therefore \quad \cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta$ is one of the qth roots of $(\cos \theta + i \sin \theta)^p$

i.e., $\cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta$ is one of the values of $(\cos \theta + i \sin \theta)^p$

Thus, if n is a fraction, positive or negative, then one of the values of $(\cos \theta + i \sin \theta)^n$ is $\cos n\theta + i \sin n\theta$.

Note 1. It may be noted that if n is integral, then $(\cos \theta + i \sin \theta)^n$ has only one value and this value is $\cos n\theta + i \sin n\theta$. On the other hand if n is fractional, then $(\cos \theta + i \sin \theta)^n$ has several values and one of its value is $\cos n\theta + i \sin n\theta$.

2. De-Moivre's theorem holds for all values of n and θ, real or complex, but we have proved it only for real θ and rational n.

Cor. If n is integral, $(\cos \theta - i \sin \theta)^n = \{\cos (-\theta) + i \sin (-\theta)\}^n$

$$= \cos (-n\theta) + i \sin (-n\theta) = \cos n\theta - i \sin n\theta$$

If n is fractional, one of the values of

$$(\cos \theta - i \sin \theta)^n \text{ is } \cos n\theta - i \sin n\theta).$$

Example 11.4.1. Simplify $\dfrac{(\cos 3\theta + i \sin 3\theta)^5 \ (\cos\theta - i \sin\theta)^3}{(\cos 5\theta + i \sin 5\theta)^7 (\cos 2\theta - i \sin 2\theta)^5}$.

Solution. The given expression

$$\frac{\{(\cos\theta + i \sin\theta)^3\}^5 \ \{(\cos\theta + i \sin\theta)^{-1}\}^3}{\{(\cos\theta + i \sin\theta)^5\}^7 \ \{(\cos\theta + i \sin\theta)^{-2}\}^5}$$

$$= \frac{(\cos\theta + \sin\theta)^{15} (\cos\theta + i \sin\theta)^{-3}}{(\cos\theta + i \sin\theta)^{35} \ (\cos\theta + i \sin\theta)^{-10}}$$

$$= (\cos \theta + i \sin \theta)^{15-3-35+10} = (\cos \theta + i \sin \theta)^{-13}$$

$$= \cos 13\theta - i \sin 13\theta.$$

Example 11.4.2. Simplify $\dfrac{\left(\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right)^{-\frac{11}{2}}}{\left(\cos\dfrac{\pi}{6} - i\sin\dfrac{\pi}{6}\right)^{\frac{1}{2}}}$

Solution. The given expression is

$$= \frac{\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^{-\frac{11}{2}}}{\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^{\frac{1}{2}}}$$

$$= \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^{\frac{-11}{2}-\frac{1}{2}}$$

$$= \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^{-6}$$

$$= \cos\pi - i\sin\pi$$

$$= -1$$

Example 11.4.3. Simplify the following :

(*i*) $\dfrac{(\cos\theta + i\sin\theta)^6 (\cos 3\theta + i\sin 3\theta)^8}{(\cos 5\theta + i\sin 5\theta)^4 (\cos 2\theta + i\sin 2\theta)^7}$ (*ii*) $(\sin\theta + i\cos\theta)^{-10}$.

Solution. (*i*) $\dfrac{(\cos\theta + i\sin\theta)^6 (\cos 3\theta + i\sin 3\theta)^8}{(\cos 5\theta + i\sin 5\theta)^4 (\cos 2\theta + i\sin 2\theta)^7}$

$$= \frac{(\cos 6\theta + i\sin 6\theta)(\cos 24\theta + i\sin 24\theta)}{(\cos 20\theta + i\sin 20\theta)(\cos 14\theta + i\sin 14\theta)}$$ [Using De-Moivre's Theorem]

$$= \frac{\text{cis}\,6\theta\,.\,\text{cis}\,24\theta}{\text{cis}\;20\theta\,.\,\text{cis}\;14\theta} \qquad [\because \quad \cos\alpha + i\sin\alpha = \text{cis}\,\alpha\,]$$

$$= \frac{\text{cis}\,(6\theta + 24\theta)}{\text{cis}\;(20\theta + 14\theta)} \qquad [\because \quad \text{cis}\,\alpha\;\text{cis}\,\beta = \text{cis}\,(\alpha + \beta)\,]$$

$$= \frac{\text{cis}\,30\theta}{\text{cis}\;34\theta} = \text{cis}\;(30\theta - 34\theta) \qquad \left[\because \quad \frac{\text{cis}\,\alpha}{\text{cis}\;\beta} = \cos\;(\alpha - \beta)\right]$$

$$= \text{cis}\;(-4\theta)$$

$$= \cos 4\theta - i\sin 4\theta \qquad [\because \quad \text{cis}\,(-\alpha) = \cos\alpha - i\sin\alpha\,]$$

(*ii*) $(\sin\theta + i\cos\theta)^{-10}$

$$= \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^{-10}$$

$$= \cos\left[-10\left(\frac{\pi}{2} - \theta\right)\right] + i\sin\left[-10\left(\frac{\pi}{2} - \theta\right)\right] \qquad \text{[Using De-Moivre's Theorem]}$$

$$= \cos\;(10\theta - 5\pi) + i\sin(10\theta - 5\pi)$$

$$= \cos\;(5\pi - 10\theta) - i\sin(5\pi - 10\theta)$$

$$= \cos\;(4\pi + \pi - 10\theta) - i\sin(4\pi + \pi - 10\theta)$$

$$= \cos\;(\pi - 10\theta) - i\sin(\pi - 10\theta)$$

$$= -\cos 10\theta - i\sin 10\theta.$$

Example 11.4.4. Prove that

$$\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n = \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right),$$ where n is any integer.

Solution. $$\left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right) = \frac{\sin^2\theta + \cos^2\theta + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}$$ [*Note it* *carefully*]

$$= \frac{(\sin^2\theta - i^2\cos^2\theta) + (\sin\theta + i\cos\theta)}{1 + \sin\theta - i\cos\theta} \qquad [\because\ i^2 = -1]$$

$$= \frac{(\sin\theta + i\cos\theta)\,[\sin\theta - i\cos\theta + 1]}{1 + \sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$$

$$= \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)$$

$$\therefore \quad \left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^n = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n$$

$$= \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{n\pi}{2} - n\theta\right) \qquad \text{[Using De-Moivre's Theorem]}$$

which is to prove.

Example 11.4.5. If $2\cos\theta = x + \frac{1}{x}$, prove that $2\cos r\theta = x^r + \frac{1}{x^r}$ where r is a integer.

Solution. $2\cos\theta = x + \frac{1}{x}$ gives $x^2 - 2x\cos\theta + 1 = 0$

$$\therefore \quad x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta$$

Take $x = \cos\theta + i\sin\theta \quad \Rightarrow \quad \frac{1}{x} = x^{-1} = (\cos\theta + i\sin\theta)^{-1}$

Then, $x^r + \frac{1}{x^r} = (\cos\theta + i\sin\theta)^r + (\cos\theta + i\sin\theta)^{-r}$

$$= (\cos r\theta + i\sin r\theta) + (\cos r\theta - i\sin r\theta)$$

$$= 2\cos r\theta$$

If $x = \cos\theta - i\sin\theta$, then $\frac{1}{x} = x^{-1} = (\cos\theta - i\sin\theta)^{-1}$

then $$x^r + \frac{1}{x^r} = (\cos\theta - i\sin\theta)^r + (\cos\theta - i\sin\theta)^{-r}$$

$$= (\cos r\theta - i\sin r\theta) + (\cos r\theta + i\sin r\theta)$$

$$= 2\cos r\theta.$$

11.5. EXAMINATION ORIENTED OBSERVATIONS

1. If $z = \cos\theta + i\sin\theta$, then prove that

(*i*) $z + \frac{1}{z} = 2\cos\theta$

(*ii*) $z - \frac{1}{z} = 2i\sin\theta$

(*iii*) $z^n + \frac{1}{z^n} = 2\cos n\theta$

(*iv*) $z^n - \frac{1}{z^n} = 2i\sin n\theta$

(*v*) $\frac{z^2 - 1}{z^2 + 1} = i\tan\theta$

2. Simplify following :

(*i*) $\frac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)}{(\cos\gamma + i\sin\gamma)(\cos\delta + i\sin\delta)}$

(*ii*) $\frac{(\cos\theta - i\sin\theta)^{10}}{(\cos\theta + i\sin\theta)^{12}}$

(*iii*) $\frac{(\cos 2\theta + i\sin 2\theta)^5 (\cos 3\theta + i\sin 3\theta)^2}{(\cos 4\theta - i\sin 4\theta)(\cos\theta + i\sin\theta)^{18}}$

3. Prove that $(\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$ and deduce

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos n\left(\frac{\pi}{2}-\theta\right)+i\sin n\left(\frac{\pi}{2}-\theta\right)$$

4. Prove that $(1+\cos\theta+i\sin\theta)^n+(1+\cos\theta-i\sin\theta)^n = 2^{n+1}\cos n\frac{\theta}{2}\cos^n\frac{\theta}{2}$.

5. If a, b are roots of $x^2 - 2x + 4 = 0$, then $a^n + b^n = 2^{n+1}\cos\frac{n\pi}{3}$.

6. Evaluate

(*i*) $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n$ (*ii*) $(1+i)^n + (1-i)^n$

7. If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$ then prove that $x_1\ x_2\ x_3$..........upto infinity $= -1$

where $r = 1, 2, 3,$.......

8. If $x+\frac{1}{x} = 2\cos\theta,\ y+\frac{1}{y} = 2\cos\phi$ then prove that

$$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta+n\phi)$$

11.6. TO FIND THE *q*TH ROOTS OF A NUMBER

Let $z = r(\cos\theta + i\sin\theta)$ be a given number.

We know that $z^{\frac{1}{q}}$ has q values. We wish to find all the values.

Now, $z^{\frac{1}{q}} = r^{\frac{1}{q}}(\cos\theta+i\sin\theta)^{\frac{1}{q}}$

$$= r^{\frac{1}{q}}\{\cos(2n\pi+\theta)+i\sin(2n\pi+\theta)\}^{\frac{1}{q}},$$

when n is any integr, positive or negative, or zero.

By De-Moivre's theorem one of the values of the right hand side is

$$r^{\frac{1}{q}}\left\{\cos\frac{2n\pi+\theta}{q}+i\sin\frac{2n\pi+\theta}{q}\right\}$$

Therefore, by giving n the values 0, 1, 2,......, $q-1$

We see that each of the quantities

$$r^{\frac{1}{q}}\left\{\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}\right\},\ r^{\frac{1}{q}}\left(\cos\frac{2\pi+\theta}{q}+i\sin\frac{2\pi+\theta}{q}\right)$$

$$\text{........},\ r^{\frac{1}{q}}\left\{\cos\frac{2(q-1)\pi+\theta}{q}+i\sin\frac{2(q-1)\pi+\theta}{q}\right\}$$

is one of the values of z.

The number of these quantities is q and they are all distinct because all the angles involed therein differ from one another by less than 2π, and no two angles differing by less than 2π have their cosines the same and also their sines the same.

Hence, these are the q values of $z^{\frac{1}{q}}$.

Note. If we give n values beyond $q-1$, we do not get any fresh value of z, the same values are repeated.

For example, putting $n = q$, we get

$$r^{\frac{1}{q}}\left\{\cos\frac{2q\pi+\theta}{q}+i\sin\frac{2q\pi+\theta}{q}\right\} = r^{\frac{1}{q}}\left\{\cos\left(2\pi+\frac{\theta}{q}\right)+i\sin\left(2\pi+\frac{\theta}{q}\right)\right\}$$

$$= r^{\frac{1}{q}}\left(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q}\right),$$

which is the same as the first value.

Note. The polar form of

$$1 = \cos 0 + i\sin 0$$

$$-1 = \cos\pi + i\sin\pi$$

$$-\,i = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$$

Example 11.6.1. Find the cube roots of unity.

Solution. $1 = \cos 0 + i \sin 0$

$$\therefore \quad (1)^{1/3} = (\cos 0 + i\sin 0)^{1/3}$$

$$= \{\cos(2n\pi + 0) + i\sin(2n\pi + 0)\}^{1/3}, \quad n = 0, 1, 2$$

$$= \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}, \; n = 0, 1, 2$$

Putting $n = 0, 1, 2$, we get for the three cube roots

$$1, \; \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$

or

$$1, \; -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\,\frac{\sqrt{3}}{2}.$$

Example 11.6.2. Find all the values of $(-1)^{1/3}$.

Solution. $-1 = \cos\pi + i\sin\pi$

$$\therefore \quad (-1)^{1/3} = (\cos\pi + i\sin\pi)^{1/3}$$

$$= \{\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)\}^{1/3}$$

$$= \cos\frac{2n\pi + \pi}{3} + i\sin\frac{2n\pi + \pi}{3}$$

Putting $n = 0, 1, 2$, we get the required values

$$\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{i\sqrt{3}}{2} = \frac{1 + i\sqrt{3}}{2}$$

$$\cos\pi + i\sin\pi = -1$$

$$\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) + i\sin\left(2\pi - \frac{\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2} = \frac{1 - i\sqrt{3}}{2}.$$

Example 11.6.3. Find all the values of $(1 - \sqrt{-3})^{1/4}$.

Solution. Let us first express

$$1 - \sqrt{-3} = 1 - i\sqrt{3}$$

For trignometric form.

Let $1 - i\sqrt{3} = r(\cos\theta + i\sin\theta)$

Then, $r\cos\theta = 1$, and $r\sin\theta = -\sqrt{3}$.

These equations give $r = 2$.

Substituting for r, we get

$$\cos\theta = \frac{1}{2}, \quad \sin\theta = -\frac{\sqrt{3}}{2}, \quad \text{which give } \theta = -\frac{\pi}{3}.$$

$$\therefore \qquad 1 - i\sqrt{3} = 2\left\{\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right\}$$

$$\therefore \qquad (1 - i\sqrt{3})^{\frac{1}{4}} = 2^{\frac{1}{4}}\left\{\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right\}^{\frac{1}{4}}$$

$$= 2^{\frac{1}{4}}\left\{\cos\left(2n\pi - \frac{\pi}{3}\right) + i\sin\left(2n\pi - \frac{\pi}{3}\right)\right\}^{\frac{1}{4}}, \; n = 0, 1, 2, 3$$

$$= 2^{\frac{1}{4}}\left\{\cos\frac{6n\pi - \pi}{12} + i\sin\frac{6n\pi - \pi}{12}\right\}, \; n = 0, 1, 2, 3$$

Putting $n = 0, 1, 2, 3$, we get required values :

$$2^{\frac{1}{4}}\left\{\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right\}, \quad 2^{\frac{1}{4}}\left\{\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right\}$$

$$2^{\frac{1}{4}}\left\{\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right\} = 2^{\frac{1}{4}}\left\{\cos\left(\pi - \frac{\pi}{12}\right) + i\sin\left(\pi - \frac{\pi}{12}\right)\right\}$$

$$= 2^{\frac{1}{4}}\left(-\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

$$= -2^{\frac{1}{4}}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)$$

$$2^{\frac{1}{4}}\left\{\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right\} = 2^{\frac{1}{4}}\left\{\cos\left(\pi + \frac{5\pi}{12}\right) + \sin\left(\pi + \frac{5\pi}{12}\right)\right\}$$

$$= 2^{\frac{1}{4}}\left(-\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}\right)$$

$$= -2^{\frac{1}{4}}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

On combining, we get four roots on

$$\pm 2^{\frac{1}{4}}\left\{\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right\}, \quad \pm 2^{\frac{1}{4}}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right).$$

Example 11.6.4. Solve the equation $x^7 + x^4 + x^3 + 1 = 0$.

Solution. The equation is $(x^4 + 1)(x^3 + 1) = 0$

Taking the first factor, we get

$$x^4 + 1 = 0$$

or $x = (-1)^{1/4} = \cos\frac{2n\pi + \pi}{4} + i\sin\frac{2n\pi + \pi}{4}$. (reference to example already solved)

Putting $n = 0, 1, 2, 3$, we get the solutions

$$\cos\frac{\pi}{4} + i\sin\frac{\pi}{4},\ \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4},\ \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4},\ \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}.$$

Taking the second factor, we get

$$x^3 + 1 = 0$$

or $$x = (-1)^{1/3} = \cos\frac{2n\pi + \pi}{3} + i\sin\frac{2n\pi + \pi}{3},\ n = 0, 1, 2$$

Putting $n = 0, 1, 2$, we get the solution

$$\cos\frac{\pi}{3} + i\sin\frac{\pi}{3},\ -1,\ \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

Hence all the roots are known.

Example11.6.5. Find *n*th root of unity and prove that the sum of their *p*th powers always vanishes unless *p* be a multiple of *n*, (*p* being and integer) and then sum is *n*.

Solution. $(1)^{1/n} = (\cos 0 + i\sin 0)^{1/n}$

$= [\cos(2r\pi + 0) + i\sin(2r\pi + 0)]^{1/n},\ r = 0, 1, 2, \ldots, n - 1$

$= \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n}$ where $r = 0, 1, 2, 3, \ldots, n - 1$

Putting $r = 0, 1, 2, \ldots, n - 1$, we get the n roots as

$$\cos 0 + i\sin 0,\ \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n},\ \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \ldots, \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}$$

The *p*th powers of the roots are

$$(1)^p,\ \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^p,\ \left(\cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}\right)^p, \ldots, \left[\cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}\right]^p$$

or $1, \cos\frac{2p\pi}{n} + i\sin\frac{2p\pi}{n}, \cos\frac{4p\pi}{n} + i\sin\frac{4p\pi}{n}, ..., \cos\frac{2p(n-1)\pi}{n} + i\sin\frac{2p(n-1)\pi}{n}$

or $1, t, t^2, ..., t^{n-1}$ where $t = \cos\frac{2p\pi}{n} + i\sin\frac{2p\pi}{n}$

Case I. Assume that p is not a multiple of n,

sum of roots $= 1 + t + t^2 + ... + t^{n-1}$

$$= 1\frac{(1-t^n)}{1-t} \qquad \left[\because \quad s_n = a\frac{(1-r^n)}{1-r}\right]$$

or $1, \frac{\cos 2p\pi}{n} + i\sin\frac{2p\pi}{n}, \left(\cos\frac{2p\pi}{n} + i\sin\frac{2p\pi}{n}\right)^2, ...\left(\cos\frac{2p\pi}{n} + i\sin\frac{2p\pi}{n}\right)^{n-1}$

$$= \frac{1}{1-t}\left[1 - \left(\cos\frac{2p\pi}{n} + i\sin\frac{2p\pi}{n}\right)^n\right]$$

$$= \frac{1}{1-t}[1 - (\cos 2p\pi + i\sin 2p\pi)]$$

$$= \frac{1}{1-t}[1 - (1 + i\,0)] = 0.$$

Case II. Assume that p is a multiple of n. Sum of roots $= 1 + t + t^2 + ... + t^{n-}$
1

$$1 + \left(\cos\frac{2p\pi}{n} + i\sin\frac{2p\pi}{n}\right) + \left(\cos\frac{4p\pi}{n} + i\sin\frac{4p\pi}{n}\right) + ...$$

$$+ \left[\cos\frac{2p(n-1)\pi}{n} + i\sin\frac{2p(n-1)\pi}{n}\right]$$

Take $p = kn$, where k is an integer.

$$= 1 + (\cos 2k\pi + i\sin 2k\pi) + (\cos 4k\pi + i\sin 4k\pi) + ...$$

$$+ [\cos 2k(n-1)\pi + i\sin 2k(n-1)\pi)]$$

$= 1 + (1 + 0) + (1 + 0) + ... + (1 + 0)$

$= n.$

Example 11.6.6. Determine the nine roots of $x^9 - 1 = 0$ by De-Moivre's Theorem and point out which of these roots satisfy $x^3 - 1 = 0$.

Solution. The given equation is $x^9 - 1 = 0$

$\Rightarrow \quad x^9 = 1$

$\Rightarrow \quad x = (1)^{1/9} = [\cos 0 + i \sin 0]^{1/9}$

$\Rightarrow \quad x = [\cos (2k\pi + 0) + i \sin (2k\pi + 0)]^{1/9}, \; k = 0, 1, 2, 3, 4, 5, 6, 7, 8$

$\Rightarrow \quad x = \cos \dfrac{2k\pi}{9} + i \sin \dfrac{2k\pi}{9}, \quad k = 0, 1, 2, 3, ... 8$

The roots of equation (1) are

$$\cos 0 + i \sin 0, \; \cos \frac{2\pi}{9} + i \sin \frac{2\pi}{9},$$

$$\cos \frac{4\pi}{9} + i \sin \frac{4\pi}{9}, \; \cos \frac{6\pi}{9} + i \sin \frac{6\pi}{9}$$

$$\cos \frac{8\pi}{9} + i \sin \frac{8\pi}{9}, \; \cos \frac{10\pi}{9} + i \sin \frac{10\pi}{9},$$

$$\cos \frac{12\pi}{9} + i \sin \frac{12\pi}{9}, \; \cos \frac{14\pi}{9} + i \sin \frac{14\pi}{9}$$

$$\cos \frac{16\pi}{9} + i \sin \frac{16\pi}{9}$$

i.e. $1, \; \text{cis} \dfrac{2\pi}{9}, \; \text{cis} \dfrac{4\pi}{9}, \; \text{cis} \dfrac{2\pi}{3}, \; \text{cis} \dfrac{8\pi}{9}, \; \text{cis} \dfrac{10\pi}{9}, \; \text{cis} \dfrac{4\pi}{3}, \; \text{cis} \dfrac{14\pi}{9}, \; \text{cis} \dfrac{16\pi}{9}$

The second given equation is $x^3 - 1 = 0$ ...(2)

$\Rightarrow \quad x^3 = 1 \quad \Rightarrow \quad x = (1)^{1/3} = (\cos 0 + i \sin 0)^{1/3}$

$= (\cos 2k\pi + i \sin 2k\pi)^{1/3}, \; k = 0, 1, 2$

$\Rightarrow \quad = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3} \quad$ where $k = 0, 1, 2$

Putting $\quad k = 0, \; x = \cos 0 + i \sin 0 = 1$

$$k = 1, \; x = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \text{cis}\frac{2\pi}{3}$$

$$k = 2, \; x = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \text{cis}\frac{4\pi}{3}$$

Hence the roots of equation (1) are

$$\text{cis}\frac{2k\pi}{9}, \quad k = 0, 1, 2, 3, ... 8$$

and common roots of equations (1) and (2) are

$$1, \; \text{cis}\frac{2\pi}{3}, \; \text{cis}\frac{4\pi}{3}$$

Example 11.6.7. If α is a non-real root nth roots of 1, show that

$$1 + \alpha + \alpha^2 + ... + \alpha^{n-1} = 0.$$

Solution. Let $z = 1 = \cos 0 + i \sin 0$

$$z^{1/n} = 1^{1/n} = (\cos 0 + i \sin 0)^{1/n}$$

$$= [\cos (0 + 2k\pi) + i \sin (0 + 2k\pi)]^{1/n}, \; k = 0, 1, 2, ..., (n-1)$$

$$= \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad \text{where } k = 0, 1, 2, ..., (n-1).$$

Let $\alpha = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$ such that α is non-real roots of unity.

Hence $1 - \alpha = 1 - \left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)$ is a non-zero number

Now L.H.S.

$$1 + \alpha + \alpha^2 + ... + \alpha^{n-1} = \frac{1.(1-\alpha^n)}{1-\alpha} \qquad \left[\text{Sum of } n \text{ terms of G.P.} = a\,\frac{(1-r^n)}{1-r}\right]$$

$$\frac{1-\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)^n}{\text{a non-zero number}} = \frac{1-(\cos 2k\pi + i\sin 2k\pi)}{\text{a non-zero no.}}$$

$$= \frac{1-(1+0i)}{\text{a non-zero no.}}$$

$$= 0 \quad \text{R.H.S.}$$

11.7. EXAMINATION ORIENTED EXERCISE

1. Find cube root of unity.

2. Evaluate the following :

(*i*) $(1+i)^{1/6}$ (*ii*) $(\sqrt{3}+i)^{1/3}$

3. Find the values of $(-i)^{1/6}$.

4. Find all the values of

(*i*) $(1-i\sqrt{3})^{2/3}$ (*ii*) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{1}{4}}$ (*iii*) $(-16i)^{1/4}$

5. Find the continued product of the four values of $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{3}{4}}$.

6. Find the four fourth roots of $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$.

7. Find all the values of $\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{\frac{3}{4}}$, and show that their continued product is unity.

8. (*i*) Show that the *n*th roots of unity form a G.P.

(*ii*) How many of the *n, n*th roots of unity are real ?

9. Find the five fifth roots of unity & prove that the sum of their *n*th power always vanishes unless *n* be a multiple of 5, *n* being an integer, and when *n* is a multiple of 5, the sum is 5.

10. Find all the roots of equation

(*i*) $x^7 + 1 = 0$ (*ii*) $x^9 - x^5 + x^4 - 1 = 0$

(*iii*) $x^4 + x^3 + x^2 + x + 1 = 0$

11. Show that the roots of equation

$$(x-1)^n = x^n,\ n \in Z_+,\ \text{are}\ \frac{1}{2}\left(1 + i\cot\frac{\pi h}{n}\right),\quad \text{where } h = 0, 1, \ldots\ldots., n-1.$$

12. Solve $x^{12} - 1 = 0$ and find which of the roots satisfy the equation $x^4 + x^2 + 1 = 0$.

13. Show that roots of equation

$$(5 + x)^5 - (5 - x)^5 = 0 \quad \text{are} \quad 5i \text{ on } \frac{k\pi}{5},\ \text{where } k = 0, 1, 2, 3, 4.$$

14. Use De-Moivre's theorem to solve

(*i*) $x^4 - x^3 + x^2 - x + 1 = 0$ (*ii*) $x^7 + x^4 + x^3 + 1 = 0$.

11.8. APPLICATION OF DE-MOIVRE'S THEOREM

Now we discuss some applications of De-Moivre's theorem.

I. Trignometric ratios of Multiple angles.

By the use of De-Moivre's theorem we can obtain the expansion of $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos\theta$, $\sin\theta$, when n is a positive integer. Also, we can obtain $\tan n\theta$ in terms of powers of $\tan\theta$.

Now, $\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n$, by De-Moivre's theorem.

Expanding the R.H.S. by Binomial theorem, we get

$$\cos n\theta + i\sin n\theta = \cos^n\theta + {}^nC_1 \cos^{n-1}\theta\,(i\sin\theta) + {}^nC_2\cos^{n-2}\theta \\ + (i\sin\theta)^2 + \ldots\ldots\ldots. + (i\sin\theta)^n$$

$$= \{\cos^n \theta - {}^nC_2 \cos^{n-2} \theta \sin^2 \theta + ...\}$$

$$+ i \{{}^nC_1 \cos^{n-1} \theta \sin \theta - {}^nC_3 \cos^{n-3} \theta \sin^3 \theta + ... + i^{n-1} (\sin^n \theta)\}$$

Hence, equating real and imaginary parts, we get

$$\cos n\theta = \cos {}^n\theta - {}^nC_2 \cos^{n-2} \theta \sin^2 \theta + \quad ...(1)$$

and

$$\sin n\theta = {}^nC_1 \cos^{n-1} \theta \sin \theta - {}^nC_3 \cos^{n-3} \theta \sin^3 \theta + \quad ...(2)$$

Each series continues till the co-efficients vanish.

From equation (1) and (2), we have, by division,

$$\tan n\theta = \frac{{}^nC_1 \cos^{n-1} \theta \sin\theta - {}^nC_2 \cos^{n-3} \theta \sin^3 \theta + ...}{\cos^n \theta - {}^nC_2 \cos^{n-2} \theta \sin^2 \theta + ...}$$

Dividing the numerator and the denominator of the right-hand side by $\cos^n \theta$, we get

$$\tan n\theta = \frac{n \tan\theta - {}^nC_3 \tan^3 \theta + ...}{1 - {}^nC_2 \tan^2 \theta + ...} \quad ...(3)$$

Cor. Putting $n = 2, 3$, we get

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} \quad \text{and} \quad \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

Note. We expanded $(\cos \theta + i \sin \theta)^n$ by Binomial Theorem.

Is this expansion valid ? Yes, it is valid.

The Proof of the Binomial theorem.

$$(x + a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + ... + a^n$$

where x and a are real and n is a positive integer, depends only on the ordinary laws of Algebra.

Complex numbers also obey these laws. Hence the theorem holds even when x and a are complex numbers.

Example 11.8.1. Expand (*i*) cos 80 in descending powers of cos θ.

(*ii*) $\frac{\sin 8\theta}{\cos\theta}$ in ascending powers of sin θ.

Solution. (*i*) We can write by De-Moivre's theorem,

$$\cos 8\theta + i \sin 8\theta = (\cos \theta + i \sin \theta)^8 \quad ...(1)$$

Expand the R.H.S. of (1) by Binomial Theorem

$$\cos 8\theta + i \sin 8\theta = \cos^8 \theta + {}^8C_1 \cos^7 \theta (i \sin \theta) + {}^8C_2 \cos^6 \theta (i \sin \theta)^2$$

$$+ {}^8C_3 \cos^5 \theta (i \sin \theta)^3 + {}^8C_4 \cos^4 \theta (i \sin \theta)^4 + {}^8C_5 \cos^3 \theta (i \sin \theta)^5$$

$$+ {}^8C_6 \cos^2 \theta (i \sin \theta)^6 + {}^8C_7 \cos \theta (i \sin \theta)^7 + {}^8C_8 (i \sin \theta)^8$$

$$= \cos^8 \theta + 8 \cos^7 \theta (i \sin \theta) - 28 \cos^6 \theta \sin^2 \theta + i (56) \cos^5 \theta \sin^3 \theta$$

$$+ 70 \cos^4 \theta \sin^4 \theta + i (56) \cos^3 \theta \sin^5 \theta - 28 \cos^2 \theta \sin^6 \theta - 8 i \cos \theta \sin^7 \theta + \sin^8 \theta$$

$$\cos 8\theta + i \sin 8\theta = (\cos^8 \theta - 28 \cos^6 \theta \sin^2 \theta + 70 \cos^4 \theta \sin^4 \theta$$

$$- 28 \cos^2 \theta \sin^6 \theta + \sin^8 \theta) + i (8 \cos^7 \theta \sin \theta - 56 \cos^5 \theta \sin^3 \theta$$

$$+ 56 \cos^3 \theta \sin^5 \theta - 8 \cos \theta \sin^7 \theta) \quad ...(2)$$

Equating real and imaginary parts, we get

$$\cos 8\theta = \cos^8 \theta - 28 \cos^6 \theta \sin^2 \theta + 70 \cos^4 \theta \sin^4 \theta - 28 \cos^2 \theta \sin^6 \theta + \sin^8 \theta \quad ...(3)$$

and $\sin 8\theta = 8 \cos^7 \theta \sin \theta - 56 \cos^5 \theta \sin^3 \theta + 56 \cos^3 \theta \sin^5 \theta - 8 \cos \theta \sin^7 \theta$ ...(4)

Putting in R.H.S. of (3), $\sin^2 \theta = 1 \cos^2 \theta$, we have

$$\cos 8\theta = \cos^8 \theta - 28 \cos^6 \theta (1 - \cos^2 \theta) + 70 \cos^4 \theta (1 - \cos^2 \theta)^2$$

$$- 28 \cos^2 \theta (1 - \cos^2 \theta)^3 + (1 - \cos^2 \theta)^4$$

$$= \cos^8 \theta - 28 \cos^6 \theta (1 - \cos^2 \theta) + 70 \cos^4 \theta (1 - 2 \cos^2 \theta + \cos^4 \theta)$$

$$- 28 \cos^2 \theta (1 - 3 \cos^2 \theta + 3 \cos^4 \theta - \cos^6 \theta)$$

$$- (1 - 4 \cos^2 \theta + 6 \cos^4 \theta - 4 \cos^6 \theta + \cos^8 \theta)$$

$$\cos 8\theta = 128 \cos^8 \theta - 256 \cos^6 \theta + 160 \cos^4 \theta - 32 \cos^2 \theta + 1 \quad ...(5)$$

(*i*) Dividing both sides of (4) by $\cos \theta$, we obtain

$$\frac{\sin 8\theta}{\cos\theta} = 8 \cos^6 \theta \sin\theta - 56\cos^4 \theta \sin^3 \theta + 56 \cos^2 \theta \sin^5 \theta - 8 \sin^7 \theta \quad ...(6)$$

Putting $\cos^2 \theta = 1 - \sin^2 \theta$ in R.H.S. of (6), we get

$$\frac{\sin 8\theta}{\cos\theta} = 8\,(1-\sin^2\theta)^3 \sin\theta - 56\,(1-\sin^2\theta)^2 \sin^3\theta + 56(1-\sin^2\theta)\sin^5\theta - 8\sin^7\theta$$

$$= 8\,(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)\sin\theta$$

$$- 56\,(1 - 2\sin^2\theta + \sin^4\theta)\sin^3\theta + 56\,(1-\sin^2\theta)\sin^5\theta - 8\sin^7\theta$$

$$\frac{\sin 8\theta}{\cos\theta} = 8\sin\theta - 80\sin^3\theta + 192\sin^5\theta - 128\sin^7\theta.$$

11.9. EXAMINATION ORIENTED EXERCISE

Prove that

1. $\cos 3\theta = -3\cos\theta + 4\cos^3\theta$

2. $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

3. $\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^2\theta$

4. $\sin 3\theta = 3\sin\theta - 4\sin\theta$

5. $\frac{\sin 6\theta}{\sin\theta} = 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta$

6. $\frac{\sin 7\theta}{\sin\theta} = 1 - 56\sin^2\theta + 112\cos^4\theta - 64\sin^6\theta$

Write down, in terms of $\tan\theta$, the values of

7. $\tan 4\theta$

8. $\tan 5\theta$.

11.10. PASCAL'S RULE FOR WRITING THE BINOMIAL COEFFICIENTS

1. The series of coefficients in successive powers of $\left(x + \frac{1}{x}\right)$ beginning with index 1 are as follows :

$\left(x + \frac{1}{x}\right)^1$ 1 1

$\left(x+\frac{1}{x}\right)^2$	1	2	1						
$\left(x+\frac{1}{x}\right)^3$	1	3	3	1					
$\left(x+\frac{1}{x}\right)^4$	1	4	6	4	1				
$\left(x+\frac{1}{x}\right)^5$	1	5	10	10	5	1			
$\left(x+\frac{1}{x}\right)^6$	1	6	15	20	15	6	1		
$\left(x+\frac{1}{x}\right)^7$	1	7	21	35	35	21	7	1	
$\left(x+\frac{1}{x}\right)^8$	1	8	28	56	70	56	28	8	1 etc.

2. The series of coefficients in successive powers of $\left(x-\frac{1}{x}\right)$ beginning with index 1 ar as follows :

$\left(x-\frac{1}{x}\right)^1$	1	–1			
$\left(x-\frac{1}{x}\right)^2$	1	–2	1		
$\left(x-\frac{1}{x}\right)^3$	1	–3	3	–1	
$\left(x-\frac{1}{x}\right)^4$	1	–4	6	–4	1

$\left(x-\frac{1}{x}\right)^5$	1	–5	10	–10	5	–1			
$\left(x-\frac{1}{x}\right)^6$	1	–6	15	–20	15	–6	1		
$\left(x-\frac{1}{x}\right)^7$	1	–7	21	–35	35	–21	7	–1	
$\left(x-\frac{1}{x}\right)^8$	1	–8	28	–56	70	–56	28	–8	1

etc.

11.10.1. To express $\sin^n \theta$, $\cos^n \theta$ in terms of sines and cosines of multiple of θ, when n is a positive integer.

$\text{Cos}^n\ \theta$.

Let $x = \cos\theta + i\sin\theta$.

Then $\frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore \quad x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

and $\frac{1}{x^n} = (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

$\therefore \quad x + \frac{1}{x} = 2\cos\theta, \quad x - \frac{1}{x} = 2i\sin\theta$

$$x^n + \frac{1}{x^n} = 2\cos n\theta, \quad x^n - \frac{1}{x^n} = 2i\sin n\theta.$$

Hence, $(2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n$

$$2^n\cos^n\theta = x^n + {}^nC_1x^{n-1}\cdot\frac{1}{x} + {}^nC_2x^{n-2}\cdot\frac{1}{x^2} + \ldots + {}^nC_{n-2}x^2\cdot\frac{1}{x^{n-2}} + {}^nC_{n-1}x\cdot\frac{1}{x^{n-1}} + \frac{1}{x^n}$$

$$= \left(x^n + \frac{1}{x^n}\right) + {}^nC_1\left(x^n + \frac{1}{x^{n-1}}\right) + {}^nC_2\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \text{........},$$

pairing terms with equal co-efficients

$$= 2 \cos n\theta + n \,\big(2\cos(n-1)\theta\big)\, \theta + \frac{n(n-1)}{2!}\,\big(2\cos(n-2)\theta\big) + \text{.......}$$

or $\quad 2^{n-1} \cos^n \theta = \cos n\theta + n \cos (n-2)\, \theta + \dfrac{n(n-1)}{2!} \cos (n-2)\, \theta + \text{.......}$

Example 11.10.2. Express $\cos^8 \theta$ in a series of cosines of multiples of θ.

(*J.U. 1988, 93*)

Solution. $(2\cos\theta)^8 = \left(x + \dfrac{1}{x}\right)^8$

$$2^8 \cos^8 \theta = x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + \frac{56}{x^2} + \frac{28}{x^4} + \frac{8}{x^6} + \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) + 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) + 56\left(x^2 + \frac{1}{x^2}\right) + 70$$

$= 2 \cos 8\theta + 8\,(2 \cos 6\theta) + 28\,(2 \cos 4\theta) + 56\,(2 \cos 2\theta) + 70.$

$\therefore \quad 2^7 \cos^8 \theta = \cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 30.$

11.11. EXAMINATION ORIENTED EXERCISE

Prove the following :

1. $2^6 \cos^7 \theta = \cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta.$
2. $2^8 \cos^9 \theta = (\cos 9\theta + 9 \cos 7\theta + 36 \cos 5\theta + 84 \cos 3\theta + 126 \cos \theta)$
3. $2^7 \sin^8 \theta = (\cos 8\theta - 8 \cos 6\theta + 28 \cos 4\theta - 56 \cos 3\theta + 35)$
4. $2^5 \sin^6 \theta \cos^2 \theta = \cos 6\theta - 2 \cos 4\theta - \cos 2\theta + 2$
5. $2^6 \sin^3 \theta \cos^4 \theta = \sin 7\theta - 3 \sin 5\theta + \sin 3\theta + 5 \sin \theta$

11.12. SUGGESTED READING

Students are advised to go through following references for details.

11.13. REFERENCE

(1) Functions of a Complex Variables by Goyal and Gupta, Pragati Prakashan, Meerut.

(2) Titu Andreescu and Dorin Andrica, *Complex Numbers from A to Z,* Birkhauser, 2006.

(3) A text Book of Real and Complex Analysis by Sunil Gupta, Udhay Banu, Ashok Kumar, Narinder Sharma, Malhotra Brothers, Pacca Danga, Jammu.

(4) James Ward Brown and Ruel V. Churchill, *Complex Variables and Applications,* 8th Ed., McGraw – Hill International Edition, 2009.

11.14. MODEL TEST PAPER

1. (*a*) Prove that n – nth roots of unity form a series in G.P.

(*b*) Expand $\sin^9 \theta$ in series of sines of multiples of θ. (*J.U. 1995*)

2. (*a*) Prove that $(1 + i)^n + (1 - i)^n = 2^{\frac{n}{2}+1} \cos\left(\frac{n\pi}{4}\right)$

(*b*) $\left(\dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = \cos\left(n\left(\dfrac{\pi}{2} - \theta\right)\right) + i\sin\left(n\left(\dfrac{\pi}{2} - \theta\right)\right)$ (*J.U. 1995*)

3. (*a*) Prove that seventh roots of unity form a series in G.P.

(*b*) Prove that

$$(1 + \cos\theta + i\sin\theta)^n + (1 + \cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2}$$ (*J.U. 1995*)

4. (*a*) Prove that $(a + ib)^{\frac{m}{n}} + (a - ib)^{\frac{m}{n}} = 2\,(a^2 + b^2)^{\frac{m}{2n}} \cos\left(\frac{m}{n}\tan^{-1}\frac{b}{a}\right)$

(*b*) Find all the values of $(1 - i)^{1/3}$. (*J.U. 1995*)

5. (*a*) Prove that $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$.

(*b*) Expand $\sin^8 \theta$ in a series of cosines of multiples of θ.

6. (*a*) If $x + \dfrac{1}{x} = 2\cos\theta$, then prove that $x^4 + \dfrac{1}{x^4} = 2\cos 4\theta$

(*b*) Find all the values of $(-1)^{\frac{1}{4}}$. (*J.U. 1994*)

7. (*a*) Prove that $\left(\dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = \cos\left(\dfrac{n\pi}{2} - n\theta\right) + i\sin\left(\dfrac{n\pi}{2} - n\theta\right)$

COMPLEX FUNCTIONS

12.1. Introduction : In this lesson the concept of Functions of Complex Variables particularly exponential and trigonometric functions are discussed.

12.2 Objectives : Objective of studying this lesson is to explain the behavour of exponential and trigonometric functions when they defined on complex domains.

12.3. FUNCTION OF COMPLEX VARIABLE3

In elementary calculus we introduced real-valued functions of real variables. That is, we discussed the function $y = f(x)$, where x takes only real values and the corresponding values of y are also real.

In particular, we defined the trignometric functions sin x, cos x, etc. the exponential function e^x, and the logarithmic function log x.

Now we define these functions for complex z, *i.e.* z is allowed to take complex values and the corresponding values of w are also permitted to be complex.

Let us take an example of a complex valued function of a complex variable.

Consider $w = z^2$, where $z = x + iy$.

Thus, $w = (x + iy)^2 = (x^2 - y^2) + 2\ ixy$

If we give any value of to z, say we put $z = 3 + 4i$, the corresponding value of w is $w = (9 - 16) + 2i\ 3.4 = -7 + 24i$, which is also complex.

Thus, w is a complex valued function of a complex variable z.

12.3.1. The exponential function e^z.

We know that

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots\ldots \qquad \ldots(1)$$

where $x \in R$ &

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + ... \qquad ...(2)$$

The expression (1) is called exponential function of x &

$$\left(1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + ... + \frac{1}{\lfloor n} + ...\right)^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + ... \qquad ...(3)$$

We know a real number is a particular case of complex number. Therefore we define exponential function of a complex quantity $z \in C$ & write it as $E(z)$ or exp. (z) or e^z *i.e.*

$$e^z = \text{exp. } (z) = E(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ... \qquad ...(4)$$

Some authors define e^z, $z = x + iy$, $x, y \in R$ as

$$e^z = e^{x+iy} = e^x . e^{iy} = e^x (\cos y + i \sin y)$$

12.3.2. Properties of exponential function e^z.

Property 1. exp. (z_1) exp (z_2) = exp $(z_1 + z_2)$

Proof. By definition of exponential functions of complex quantity

$$\exp (z_1) = 1 + z_1 + \frac{z_1^2}{\lfloor 2} + \frac{z_1^3}{\lfloor 3} + \frac{z_1^4}{\lfloor 4} + ...$$

$$\exp (z_2) = 1 + z_2 + \frac{z_2^2}{\lfloor 2} + \frac{z_2^3}{\lfloor 3} + \frac{z_2^4}{\lfloor 4} + ...$$

Since the above series are convergent or have finite and unique sum, let them get multiplied.

Grouping together the terms of the same degree in z_1 & z_2 we have

$$\text{exp. } (z_1) \exp (z_2) = 1 + (z_1 + z_2) + \left(\frac{z_1^2}{\lfloor 2} + z_1 z_2 + \frac{z_2^2}{\lfloor 2}\right) + ...$$

$$= 1 + (z_1 + z_2) + \frac{(z_1 + z_2)^2}{\lfloor 2} + \frac{(z_1 + z_2)^3}{\lfloor 3} + ...$$

$$= \text{exp. } (z_1 + z_2)$$

Alternate method.

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

Then, $e^{z_1} \cdot e^{z_2} = e^{x_1+iy_1} \cdot e^{x_2+iy_2}$

$$= e^{x_1}.(\cos y_1 + i\sin y_1).e^{x_2}.\cos(y_2 + i\sin y_2)$$

$$= e^{x_1+x_2} \cdot \{\cos(y_1 + y_2) + i\sin(y_1 + y_2)\}$$

$$= e^{(x_1+x_2)+i(y_1+y_2)}$$

$$= e^{z_1+z_2}$$

Cor 1. The result may be generalized as :

$$e^{z_1} \cdot e^{z_2} ....e^{z_n} = e^{(z_1+z_2+.....+z_n)}$$

Putting $z_1 = z_2 .... = z_n = z,$ we get

$(e^z)^n = e^{nz}.$

Thus, if n be a positive integer, then $(e^x)^n = e^{nz}$.

Cor 2. $e^{z_1-z_2} \cdot e^{z_2} = e^{(z_1-z_2)+z_2} = e^{z_1}$

$\therefore$ $$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

Cor 3. $e^z.e^{-z} = e^{z+(-z)} = e^0 = 1$

$\therefore$ $$e^{-z} = \frac{1}{e^z}$$

12.3.3. Theorem : e^z is periodic with period $2\pi i$.

Proof. $e^{z+2\pi i} = e^{(x+iy)+2\pi i}$

$$= e^{x+i(y+2\pi i)}$$

$$= e^x [\cos (y + 2\pi) + i \sin (y + 2\pi)]$$

$$= e^x. (\cos y + i \sin y)$$

$$= e^{x+iy} = e^z.$$

$\therefore$ e^z is periodic with period $2\pi i$.

12.3.4. Euler's Exponential values for sin *x* and cos *x*.

We know that

$$e^{x+iy} = e^x (\cos y + i \sin y)$$

Putting $x = 0$, we get

$$e^{iy} = \cos y + i \sin y \qquad ...(1)$$

Changing y to $-y$, we get

$$e^{-iy} = \cos y - i \sin y \qquad ...(2)$$

Adding, we get

$$e^{iy} + e^{-iy} = 2 \cos y$$

or

$$\cos y = \frac{1}{2}(e^{iy} + e^{-iy}) \qquad ...(3)$$

Similarly, subtracting we get

$$e^{iy} - e^{-iy} = 2 i \sin y$$

or

$$\sin y = \frac{1}{2i}(e^{iy} - e^{-iy}) \qquad ...(4)$$

The formulae (3) and (4) express the sine and cosine of a real variable in terms of the exponential function and are due to the mathematician Euler.

12.4. THE COMPLEX CIRCULAR FUNCTIONS sin *z*, cos *z*.

Again we want to define sin z and cos z in such a manner that they may obey the same laws as sin x and cos x.

By Euler's formula.

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \quad \text{and} \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

We take these as the definitions of cos z and sin z.

Thus,

12.4.1. Definition. $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

Note. The other circular functions are defined as in the case of real ariable.

Thus, $\tan z = \dfrac{\sin z}{\cos z}, \cot z = \dfrac{\cos z}{\sin z}, \sec z = \dfrac{1}{\cos z}$, and $\operatorname{cosec} z = \dfrac{1}{\sin z}$

12.4.2. Remark. We have $\cos z = \dfrac{1}{2}(e^{iz} + e^{-iz})$ and $\sin z = \dfrac{1}{2i}(e^{iz} + e^{-iz})$

There are two equation given

$$e^{iz} = \cos z + i \sin z.$$

and $$e^{-iz} = \cos z - i \sin z.$$

2.4.3. Example. Prove that

$$\{\sin (\alpha + \theta) - e^{i\alpha} \sin \theta\}^n = \sin^n\alpha . e^{-n\theta i}$$

Solution. L.H.S. $= \{(\sin \alpha \cos \theta + \cos \alpha \sin \theta) - (\cos \alpha + i \sin \alpha) \sin \theta]^n$

$$= (\sin \alpha \cos \theta - i \sin \alpha \sin \theta)^n$$

$$= \sin^n \alpha (\cos \theta - i \sin \theta)^n$$

$$= \sin^n \alpha . (e^{-i\theta})^n$$

$$= \sin^n \alpha .\ e^{-n\theta i}$$

12.4.4. Example. Prove that for complex z

$$\cos^2 z + \sin^2 z = 1.$$

Solution. $\cos^2 z + \sin^2 z = \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2$

$$= \frac{(e^{iz} + e^{-iz})^2}{4} - \frac{(e^{iz} - e^{-iz})^2}{4}$$

$$= \frac{1}{4} \cdot 4e^{iz}.e^{-iz} = 1.$$

12.4.5. Example. Apply the exponential values of sine and cosine to show that :

(*i*) $\sin 2z = 2 \sin z \cos z$.

(*ii*) $\cos 2z = 1 - 2 \sin^2 z = 2 \cos^2 z - 1$

(*iii*) $\cos 3z = 4\cos^3 z - 3\cos z$.

Solution. As we know $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \dfrac{e^{iz} + e^{-iz}}{2i}$

(*i*) L.H.S. $= \sin 2z = \dfrac{e^{2iz} - e^{2iz}}{2i} = \dfrac{2[(e^{iz})^2 - (e^{-iz})^2]}{4i}$

$$= 2\frac{(e^{iz} - e^{-iz})}{2i}\left(\frac{e^{iz} + e^{-iz}}{2}\right) = 2\sin z\cos z.$$

(*ii*) As $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$\therefore \quad 1 - 2\sin^2 z = 1 - 2\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$

$$= 1 - \frac{2}{4i^2}\left[e^{2iz} + e^{-2iz} - 2e^{iz}.e^{-iz}\right]$$

$$= 1 + \frac{1}{2}\left[e^{2iz} + e^{-2iz} - 2\right]$$

$$= 1 + \left(\frac{e^{2iz} + e^{-2iz}}{2}\right) - 1$$

$$= \cos 2z$$

Also $\quad 2\cos^2 z - 1 = 2\left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2 - 1$

$$= \frac{1}{2}\left[e^{2iz} + e^{-2iz} + 2e^{iz}.e^{-iz} - 2\right]$$

$$= \frac{1}{2}\left[e^{2iz} + e^{-2iz} + 2 - 2\right]$$

$$= \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z$$

(*iii*) L.H.S. $\cos 3z = \frac{e^{3iz} + e^{-3iz}}{2}$

$$= \frac{(e^{iz})^3 + (e^{-iz})^3}{2}$$

$$= \frac{1}{2}\left[(e^{iz} + e^{-iz})^3 - 3e^{iz}.e^{-iz}(e^{iz} + e^{-iz})\right]$$

$$\left[\because a^3 + b^3 = (a+b)^3 - 3ab(a+b)\right]$$

$$= 4\left(\frac{e^{iz} + e^{-iz}}{2}\right)^3 - 3\left(\frac{e^{iz} + e^{-iz}}{2}\right)$$

$$= 4\ (\cos z)^3 - 3\cos z$$

$$= 4\cos^3 z - 3\cos z = \text{R.H.S.}$$

12.5. SUGGESTED READING

The students are advised to go through following references for details.

12.6. REFERENCES

(1) Functions of a Complex Variables by Goyal and Gupta, Pragati Prakashan, Meerut.

(2) Titu Andreescu and Dorin Andrica, *Complex Numbers from A to Z,* Birkhauser, 2006.

(3) A text Book of Real and Complex Analysis by Sunil Gupta, Udhay Banu, Ashok Kumar, Narinder Sharma, Malhotra Brothers, Pacca Danga, Jammu.

(4) James Ward Brown and Ruel V. Churchill, *Complex Variables and Applications,* 8th Ed., McGraw – Hill International Edition, 2009.

12.7. MODEL TEST PAPER

1. Separate into real and imaginary parts $e^{5+\frac{1}{2}\pi i}$

2. Prove that $\sin(\alpha + n\theta) - e^{i\alpha} \sin n\theta = e^{-n\theta i}. \sin \alpha$.

Prove that for complex x.

3. $\sin(-x) = -\sin x$

4. $\cos(-x) = \cos x$.

5. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

6. $\sin 3x = 3\sin x - 4\sin^3 x$.

7. $\cos 3x = 4\cos^3 x - 3\cos x$.

8. $\sin 2x = 2\sin x \cos x$.

Prove that for complex x and y.

9. $\sin x + \sin y = 2\sin \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

10. $\cos x + \cos y = 2\cos \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$.

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FUNCTIONS OF COMPLEX VARIABLE

13.1. Introduction : In this lesson the concept of Functions of Complex Variables particularly hyperbolic, inverse hyperbolic and their relation with logarithmic functions are discussed.

13.2 Objectives : Objective of studying this lesson is to explain the properties of hyperbolic, inverse hyperbolic and their relation with logarithmic functions.

13.3. HYPERBOLIC FUNCTIONS

13.3.1. Definition. For real or complex z, $\dfrac{e^z + e^{-z}}{2}$ is called the hyperbolic cosine of z and written as cosh z.

Similarly $\dfrac{e^z - e^{-z}}{2}$ is called the hyperbolic sine of z and written as sin h z.

Thus, $\cosh z = \dfrac{e^z + e^{-z}}{2}$ and $\sinh z = \dfrac{e^z - e^{-z}}{2}$.

The hyperbolic tangent, contangent, secant and cosecant are defined terms of the hyperbolic sine and cosine and in the same manner as the ordinary tangent, cotangent, secant, and cosecant in terms of the ordinary sine and cosine.

Thus, $\tanh z = \dfrac{\sinh z}{\cosh z} = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$

$\cot h\ z = \dfrac{1}{\tanh z} = \dfrac{e^z + e^{-z}}{e^z - e^{-z}}$

$$\text{sec h } z = \frac{1}{\cosh z} = \frac{2}{e^z + e^{-z}}$$

$$\text{cosech } z = \frac{1}{\sinh z} = \frac{2}{e^z - e^{-z}}$$

Cor. $\sin h\ o = \dfrac{e^o - e^o}{2} = \dfrac{1-1}{2} = 0$

$$\cosh o = \frac{e^o + e^{-o}}{2} = \frac{1+1}{2} = 1$$

$$\tanh o = \frac{\sinh o}{\cosh o} = 0.$$

Thus, sin h o = 0, cosh o = 1, tanh o = 0.

Again,

$$\sinh(-z) = \frac{e^{-z} - e^{-(-z)}}{2} = \frac{e^{-z} - e^z}{2} = -\left(\frac{e^z - e^{-z}}{2}\right) = -\sinh z.$$

$$\cosh(-z) = \frac{e^{-z} - e^{-(-z)}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z.$$

$$\tanh(-z) = \frac{\sinh(-z)}{\cosh(-z)} = \frac{-\sinh z}{\cosh z} = -\tanh z.$$

Thus, sinh ($-z$) = –sinh z, cosh ($-z$) = cosh z, and tan h ($-z$) = –tan h z.

13.3.2. Relation between Circular and Hyperbolic Functions.

$$\sin(iz) = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{e^{-z} - e^z}{2i} = \frac{i(e^{-z} - e^z)}{2i^2}$$

$$= \frac{-i}{2}(e^{-z} - e^z) = i \cdot \frac{1}{2}(e^z - e^{-z}) = i.\sinh z.$$

Again, $\cos(iz) = \dfrac{e^{i(iz)} + e^{-i(iz)}}{2} = \dfrac{e^{-z} + e^z}{2} = \cosh z$

Finally, $\tan(iz) = \dfrac{\sin(iz)}{\cos(iz)} = \dfrac{i\sinh z}{\cosh z} = i.\tanh z.$

Thus, **sin (*iz*) = *i* sin h *z*.**

cos (*iz*) = cosh*z*,

and tan (*iz*) = *i* tan h*z*.

13.3.3. Formulae in hyperbolic functions.

Corresponding to formulae in circular functions there are formula in hyperbolic functions.

These can be obtained directly from the definitions of hyperbolic functions or from the above relations between the circular and hyperbolic functions.

13.3.4. Example. $\cosh^2 z - \sin h^2 z = 1$

Solution. $\cosh^2 z - \sinh^2 z = \left(\dfrac{e^z + e^{-z}}{2}\right)^2 - \left(\dfrac{e^z - e^{-z}}{2}\right)^2$

$$= \frac{e^{2z} + 2 + e^{-2z}}{4} - \frac{e^{2z} - 2 + e^{-2z}}{4} = 1.$$

13.3.5. Example. $\sin h(x + y) = \sin h\, x \cosh y + \cosh x + \sinh y.$

Solution. It is easier to obtain the result by the second method.

For all u and v we have

$$\sin(u + v) = \sin u \cos v + \cos u + \sin v.$$

Let $u = ix$ and $v = iy$.

We obtain

$$\sin i(x + y) = \sin ix \cos iy + \cos ix \sin iy$$

or, $i \sin h(x + y) = i \sinh x \cos h\, y + \cosh hx.i \sin h\, y$

Cancelling out *i,* we get the result.

Note. Since $\cos(ix) = \cosh x$, it follows that any general formula which is true for cosines of angle is also true if instead of cos we write cosh.

Again, since $\sin(iy) = i \sin h\, y$, it follow that $\sin^2(iy) = -\sinh^2 y$ and so any formula involving the cosines and the square of the sine of an angle is true if for cos we write cosh and for $\sin^2$ we write $\sinh^2$.

Similarly, we may prove a formula involving $\tan^2$ into another by writing $\tanh^2$ for $\tan^2$.

13.3.6. Period of the hyperbolic functions :

$\cosh z = \cos iz = \cos(-2\pi + iz)$, _ $\cos z$ is periodic with period 2π.

$= \cos i\,(2\pi\, i + z)$

$= \cos h\,(2\pi\, i + z)$

_ $\cos h\, z$ is periodic with period $2\pi i$.

Second Method

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$= \frac{1}{2}(e^z . e^{2\pi i} + e^{-z} . e^{-2\pi i}) \quad \left[\because e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 \text{Also}, e^{-2\pi i} = \frac{1}{e^{2\pi i}} = 1\right]$$

$$= \left\{e^{z+2\pi i} + e^{-(z+2\pi i)}\right\}$$

$$= \cos h\,(z + 2\pi\, i)$$

Hence, the result

Again, $i \sin h\, z = \sin(iz)$

$= \sin(-2\pi + iz)$

$= \sin i\,(2\pi i + z)$

$= i \sin h\,(2\pi i + z)$

$\therefore$ $\sinh z = \sin h\,(2\pi i + z)$

Hence, the period of $\sin h\, z$ is $2\pi i$.

Finally, $i \tan h\, z = \tan(z)$

$= \tan(-\pi + iz)$, _ $\tan z$ is periodic

$= \tan i\,(\pi i + z)$

$= i, \tan h\,(\pi i + z)$

$\therefore$ $\tan h\, z = \tan h\,(\pi i + iz)$

Hence, period of tan h z is πi.

13.3.7. Series expansions of sinh z and cosh z.

$$\cosh z = \frac{1}{2}(e^z + e^{-z})$$

$$= \frac{1}{2}\left[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...\right) + \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + ...\right)\right] \qquad ...(A)$$

$$= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + ...$$

$$\sin h\ z = \frac{1}{2}(e^z - e^{-z})$$

$$= z + \frac{z^3}{3!} + \frac{z^5}{5!} + ...$$

Note : Here again we have assumed that we can combine the terms of two infinite series in (A).

13.3.8. Example. Separate the following into real and imaginary parts :

(*a*) sin $(\alpha + i\beta)$ (*b*) tan $(\alpha + i\beta)$

(*c*) cos h $(\alpha + i\beta)$ (*d*) cot h $(\alpha + i\beta)$

Solution. (*a*) $\sin (\alpha + \beta) = \sin \alpha \cos (i\beta) + \cos \alpha \sin (i\beta)$

$$= \sin \alpha \cos h\ \beta + i \cos \alpha \sin h\ \beta.$$

(*b*) $$\tan (a + i\beta) = \frac{\sin(a + i\beta)}{\cos(\alpha + i\beta)}$$

$$= \frac{2\sin(\alpha + i\beta)\cos(\alpha - i\beta)}{2\cos(\alpha + i\beta)\cos(a - i\beta)}$$

$$= \frac{\sin 2\alpha + \sin 2i\beta}{\cos 2\alpha + \cos 2i\beta}$$

$$= \frac{\sin 2\alpha + i\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$$

Note : We express tan in terms of sin and cos.

Second Method.

Let $\tan(\alpha + i\beta) = x + iy$ ...(1)

Then, $\tan(\alpha - i\beta) = x - iy$ ...(2)

Now, adding (1) and (2), we get

$$2\alpha = \tan(\alpha + i\beta) + \tan(\alpha - i\beta)$$

$$= \frac{\sin(\alpha + i\beta)}{\cos(\alpha + i\beta)} + \frac{\sin(\alpha - i\beta)}{\cos(\alpha - i\beta)}$$

$$= \frac{\sin(\alpha + i\beta)\cos(\alpha - i\beta) + \cos(\alpha + i\beta)\sin(\alpha - i\beta)}{\cos(\alpha + i\beta)\cos(\alpha - i\beta)}$$

$$= \frac{\sin(\alpha + i\beta + \alpha - i\beta)}{\frac{1}{2}[\cos 2\alpha + \cos 2i\beta)} = \frac{2\sin 2\alpha}{\cos 2\alpha + \cosh 2\beta}$$

$$\therefore \qquad x = \frac{2\sin 2\alpha}{\cos 2\alpha + \cosh 2\beta}$$

Similarly substracting (2) from (1), we get

$$y = \frac{2\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}.$$

(*c*) $\cosh(\alpha + i\beta) = \cos i(\alpha + i\beta) = \cos(i\alpha - \beta)$

$= \cos i\alpha \cos\beta + \sin i\alpha \sin\beta$

$= \cos h\, \alpha \cos\beta + i \sin h\, \alpha \sin\beta.$

Note. We express cosh in terms of cosh.

(*d*) $i \coth(\alpha + i\beta) = \cot i(\alpha + i\beta)$ [_ $\cot iz = i \coth z$]

$= \cot(i\alpha - \beta)$

$$= \frac{\cos(i\alpha - \beta)}{\sin(i\alpha - \beta)} \cdot \frac{2\sin(i\alpha + \beta)}{2\sin(i\alpha + \beta)}$$

$$= \frac{\sin 2\alpha i + \sin 2\beta}{\cos 2\beta - \cos 2i\alpha}$$

$$= \frac{i \sin 2\alpha + \sin 2\beta}{\cos 2\beta - \cosh 2\alpha}$$

$$= \frac{i(\sin 2\alpha - i \sin 2\beta)}{\cos 2\beta - \cosh 2\alpha}$$

$$\therefore \quad \coth(\alpha + i\beta) = \frac{\sinh 2\alpha - i \sin 2\beta}{\cos 2\beta - \cosh 2\alpha}$$

13.3.9. Example. If sin (A + *i*B) = $x + iy$, prove that $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$.

Solution. $x + iy = \sin(A + iB)$

$$= \sin A \cos h B + i \cos A \sin h B$$

[By separating R.H.S. into real and imaginary parts]

∴ Equating real and imaginary parts,

$$x = \sin A \cos B \qquad ...(i)$$

$$y = \cos A \sin h B \qquad ...(ii)$$

We get the desired result by eliminating B from (*i*) and (*ii*).

From (*i*), $\cosh B = \frac{x}{\sin A}$

and from (*ii*), $\sin h B = \frac{y}{\cos A}$

$$\therefore \quad \cosh^2 B - \sinh^2 B = \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A}$$

i.e.

$$1 = \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A}.$$

(Note that to eliminate B we have made use of the formula $\cosh^2 B - \sinh^2 B = 1$).

13.3.10. Example. Show that $\left(\frac{1+\tanh x}{1-\tanh x}\right)^3 = \cosh 6x + \sinh 6x.$

Solution. L.H.S. $= \left(\frac{1+\tanh x}{1-\tanh x}\right)^3$

$$= \left[\frac{1+\frac{\sinh x}{\cosh x}}{1-\frac{\sinh x}{\cosh x}}\right]^3 = \left[\frac{\cosh x + \sinh x}{\cosh x - \sinh x}\right]^3$$

$$= \left(\frac{e^x}{e^{-x}}\right)^3 = (e^{2x})^3 = e^{6x}$$

= cosh 6x + sinh 6x = R.H.S.

13.4. EXAMINATION ORIENTED EXERCISE

1. Prove that

(*i*) cos h (α + β) = cosh α cos h β + sinh α sin hβ.

(*ii*) sinh 3x = 3 sin h x + 4 $\sinh^3 x$

(*iii*) tan 3x = $\frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$

2. tan (α + β) = $\frac{\tan\alpha + \tanh\beta}{1+\tanh\alpha + \tanh\beta}$

3. cosh (α + β) – cosh (α – β) = 2 sin hα sin hβ

4. (*i*) 2 sin h A cos B = sin h (A + B) + sinh (A – B).

(*ii*) show that log $\left[\frac{\cos(x+iy)}{\cos(x-iy)}\right]$ is purely imaginary.

5. If tan y = tan α tanh β and tan z = cot β tanh β, prove that

tan (y + z) = sinh 2 β cosec 2α.

6. If $\cosh x = \sec \theta$, prove that $\tanh^2 \dfrac{x}{2} = \tan^2 \dfrac{\theta}{2}$.

13.5. THE COMPLEX INVERSE CIRCULAR FUNCTION

13.5.1. Inverse cosine.

If $\cos (x + iy) = u + iv$, then $x + iy$ is defined as an inverse cosine of $u + iv$.

But $\cos (x + iy) = \cos [2n\pi \pm (x + iy)]$, so that $2n\pi \pm (x + iy)$ is also an inverse cosine of $u + iv$ where n is an integer including zero.

The inverse cosine of $u + iv$ is thus a many valued function. When the many-valued ness of inverse cosine is considered it is written $\cos^{-1} (u + iv)$.

The principal value of the inverse cosine of $u + iv$ is that value whose real part lies between θ and π This value is denoted by $\cos^{-1}(u + iv)$.

Thus, we write $\cos^{-1}(u + iv) = 2n\pi \pm (x + iy) = 2n\pi \pm \cos^{-1} (u + iv)$, to indicate that all the values of the inverse cosine of $(u + iv)$ are obtained from the expression $2n\pi \pm \cos^{-1}(u + iv)$, where $\cos^{-1} (u + iv)$ denotes the principal value of the inverse cosine of $u + iv$ and n is any integer, including zero.

13.5.2. Inverse sine.

If $u + iv = \sin (x + iy) = \sin [n\pi + (-1)^n (x + iy)]$, then $n\pi + (-1)^n (x + iy)$,is an inverse sine of $u + iv$. It is a many valued function and is denoted by $\sin^{-1} (u + iv)$.

Its principal value is such that its real part lies between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$.

This value is denoted by $\sin^{-1} (u + iv)$.

13.5.3. Inverse tangent.

If $u + iv = \tan (x + iy) = \tan [n\pi + (x + iy)]$, then $n\pi + (x + iy)$ is an inverse tangent of $u + iv$. It is written as $\tan^{-1}(u + iv)$.

Its principal value is such that its real part lies between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$.

Thus, $\tan^{-1}(u + iv) = n\pi + \tan^{-1}(x + iy)$

Similarly, $\cot^{-1} (u + iv) = n\pi + \cot^{-1}(x + iy)$

$\sec^{-1}(u + iv) = 2n\pi + \sec^{-1} (x + iy)$,

and $\operatorname{cosec}^{-1}(u + iv) = n\pi + (-1)^n \operatorname{cosec}^{-1}(x + iy)$.

13.5.4. Example. Separate $\tan^{-1}(\alpha + i\beta) + x + iy$

Solution. Let $\tan^{-1}(\alpha + i\beta) = x + iy$

Then, $\tan(x + iy) = \alpha + i\beta$ ...(*i*)

$\therefore$ $\tan(x - iy) = \alpha - i\beta.$ ...(*ii*)

$$\therefore \quad \tan[(x + iy) + (x + iy)] = \frac{(\alpha + i\beta) + (\alpha - i\beta)}{1 - (\alpha + i\beta)(\alpha - i\beta)}$$

or $$\tan 2x = \frac{2\alpha}{1 - \alpha^2 - \beta^2}$$

$$\therefore \quad x = \frac{1}{2}\tan^{-1}\frac{2\alpha}{1 - \alpha^2 - \beta^2}$$

Again, $$\tan(x + iy - x - iy) = \frac{(\alpha + i\beta) - (\alpha + i\beta)}{1 + (\alpha + i\beta)(\alpha - i\beta)}$$

or $$\tan(2iy) = \frac{2i\beta}{1 + \alpha^2 + \beta^2}$$

or, $$\tanh 2y = \frac{2\beta}{1 + \alpha^2 + \beta^2}$$

$$\therefore \quad y = \frac{1}{2}\tanh^{-1}\frac{2\beta}{1 + \alpha^2 + \beta^2}$$

Henec, $x + iy = \frac{1}{2}\tan^{-1}\frac{2\alpha}{1 - \alpha^2 - \beta^2} + \frac{i}{2}\tanh^{-1}\frac{2\beta}{1 + \alpha^2 + \beta^2}$.

13.6. EXAMINATION ORIENTED EXERCISE

Separate into real and imaginary parts.

1. $\sin^{-1}(\cos\theta + i\sin\theta)$s, where θ is a positive angle $< \pi$.

2. $\tan^{-1}(\cos\theta + i\sin\theta)$.

3. Prove that $\sin^{-1}(\text{cosec}\,\theta) = \left\{2n + (-1)^n\right\}\frac{\pi}{2} + i(-1)^n\cosh^{-1}(\text{cosec}\,\theta)$, when θ

lies between θ and π.

4. $\cos^{-1}(\sec\theta) = 2n\pi \pm i\cosh^{-1}(\sec\theta)$, if $\sec\theta$ is positive $= (2n+1)\pi + i\cosh^{-1}(-\sec\theta)$, if $\sec\theta$ is negative.

5. $\tan^{-1}(\cos\theta + i\sin\theta) = n\pi \pm \frac{\pi}{4} + \frac{i}{2}\tanh^{-1}(\sin\theta)$

According as $\cos\theta$ its positive or negative.

Prove that

6. If $a = ib = \sin^{-1}(\cos\theta + i\sin\theta)$, then $\cos^2 a = \sinh^2 b$.

7. If $a = ib = \cos^{-1}(\alpha + \beta)$, then $\alpha^2\sec^2 a + \beta^2\operatorname{cosec}^2 a = 1$

and $\alpha^2\operatorname{sech}^2 b + \beta^2\operatorname{cosech}^2 b = 1$.

8. Prove that $\tan^{-1}\dfrac{\tan 2\theta + \tanh 2\phi}{\tan 2\theta - \tan 2\phi} + \tan^{-1}\dfrac{\tan\theta - \tan\phi}{\tan\theta + \tanh\phi} = \tan^{-1}(\cot\theta\coth\phi)$.

13.7. INVERSE HYPERBOLIC FUNCTIONS.

If $\sinh u = z$, then u is called an inverse sinh of z and written as $\sinh^{-1} z$.

Similarly other inverse hyperbolic functions can be defined.

It can be shown that if z is real, then $\sinh^{-1}z$, $\cosh^{-1}$ $\tanh^{-1}z$, etc are single-valued. On the other hand, if z is complex, these functions are many valued.

13.7.1. Logarithmic expressions for real inverse hyperbolic functions $\sinh^{-1}x$.

Let $\sinh^{-1} x = y$

Then, $x = \sinh y = \dfrac{e^y - e^{-y}}{2} = \dfrac{e^{2y} - 1}{2e^y}$

$\therefore \quad e^{2y} - 2xe^y - 1 = 0\cdot$

Solving it as a quadratic in e^y, we get

$$e^y = x \pm \sqrt{x^2+1}\cdot$$

Since e^y is always positive, we take plus sign before the radical.

Thus, $e^y = x + \sqrt{x^2+1}$ or $y = \log\left(x + \sqrt{x^2+1}\right)$

Hence, $\sinh^{-1}x = \log\left(x+\sqrt{x^2+1}\right)$

13.7.2. For $\cosh^{-1}x$

Let $\cosh^{-1} x = y$

Then, $x = \cosh y = \dfrac{e^y + e^{-y}}{2} = \dfrac{e^{2y}+1}{2e^y}$

$\therefore \quad e^{2y} - 2xe^y + 1 = 0$

$\therefore \quad e^y = x \pm \sqrt{x^2-1} = x+\sqrt{x^2-1}, x-\sqrt{x^2-1}$

or $\quad y = \log\left(x \pm \sqrt{x^2-1}\right)$

The convention is to take plus sign before the radical.

Thus, $\quad \cosh^{-1} x = \log\left(x+\sqrt{x^2-1}\right)$

13.7.3. For $\tanh^{-1}x$

Let $\tanh^{-1}x = y$

Then, $x = \tanh y = \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$

$\therefore \quad \dfrac{1+x}{1-x} = \dfrac{2e^y}{2e^{-y}} = e^{2y} \quad$ or $\quad y = \dfrac{1}{2}\log\dfrac{1+x}{1-x}$

Thus, $\quad \tanh^{-1}x = \dfrac{1}{2}\log\dfrac{1+x}{1-x}$

13.8. COMPLETE LOGARITHMIC FUNCTIONS

Def. If $\alpha = e^x$ where α and x are real, we know that x is called the logarithm of α to the base e.

We now extend this definitions to complex quantities.

If $u = e^x$, where u and z are complex, then z is called a logarithm of u to the base e.

But $u = e^z = e^z.e^{n\pi i}$ $(\therefore\ e^{2\pi i} = \cos 2n\pi + i \sin 2n\pi = 1)$

$= e^{z+2n\pi}.$

$\therefore$ $z + 2n\pi$ is also a logarithm of u to the base e.

The logarithm of of u is thus a many-valued function. We denote this by writing log u for the general value.

13.8.1. To find all the values of Log *x*.

Let $z = x + iy = r(\cos\theta + i \sin\theta), -\pi < \theta < \pi.$

$= r\,[\cos(2n\pi + \theta) + i \sin(2n\pi + \theta)$

Where n is any integer, and r and θ satisfy the two equations $x = r\cos\theta$, $y \sin\theta$,

so that $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\dfrac{y}{x}$

Let $\log z = u + iv$

Then, $z = e^{u+iv} = e^u(\cos v + i \sin v)$

i.e. $r\,[\cos(2n\pi + \theta) + i \sin(2n\pi + \theta)] = e^u(\cos v + i \sin v)$

$\therefore$ $e^u = r$, so that $u = \log r$ and $v = 2n\pi + \theta$.

Hence, $\text{Log } z = u + iv = \log r + i\,(2n\pi + \theta)$.

Thus, $\text{Lot } z = 2n\pi + (\log r + i\theta)$, when n is any integer including zero.

Note. The value obtained by putting n equal to zero is called the principal value of Log z and is denoted by log z, so that

$\text{Log } x = 2n\pi + \log z$

Thus, $\log z = \log r + i\theta$ *i.e.* $\text{Log }(x + iy) = \log \sqrt{x^2 + y^2} + i \tan^{-1}\dfrac{y}{x}$

and $\text{Log } z = 2n\pi i + (\log r + i\theta)$

i.e. $\text{Log }(x + iy) = 2n\pi i + \left(\log\sqrt{x^2 + y^2} + i \tan^{-1}\dfrac{y}{x}\right)$

13.8.2. Laws of Logarithms

If z_1 and z_2 are any two complex numbers, then

(*i*) $\log(z_1 z_2) = \log z_1 = \log z_2$

(*ii*) $\log \dfrac{z_1}{z_2} = \log z_1 - \log z_2$

(iii) $\log z_1^n = n \log z_1$

These equations are not necessarily true for the principal value. Actually these relations express that every value of the left side is equal to some value of the right side.

13.8.3. The logarithm of a positive real number.

We have $\log (x + iy) = 2n\pi i + \left(\log\sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x}\right)$

Put $y = 0$

We get $\text{Log } x = 2n\pi i + \log x$

Thus, Log x has one real value viz. $\log x$.

which is the ordinary logarithm of x.

Hence, every positive real number has a real logarithm, which is its ordinary logarithm.

Note : It may be noted that the principal value of the logarithm of a +ve real number is equal to its ordinary logarithm.

13.8.4. The logarithm of a negative real number.

We have

Lot $(x + iy) = 2n\pi i + (\log r + i\theta)$ where $x = r \cos \theta$ $y = r \sin \theta$ $-\pi < \theta < \pi$.

Put $y = 0$ and $x = -\alpha$ where α is positive.

With these substitutions we obtain $r = \alpha$ $\sin \theta = \pi$, such that

$$\text{Log } (-\alpha) = 2n\pi i + \log \alpha + i\pi.$$

Hence, (a) Log $(-\alpha)$ has no real value,

and (b) the principal value of Log is $\log \alpha + i\pi$ *i.e.* $\log (-\alpha) = \log \alpha + \pi i$

13.8.5. Example. Find all the values of Log $(1 + i)$.

Solution. Let $1 + i = r (\cos \theta + i \sin \theta)$

Then, $r \cos \theta = 1$ and $r \cos \theta = 1$, giving $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

$\therefore$ $\text{Log } (1 + i) = 2n\pi i + \log r + i\theta$

$$= 2n\pi i + \log \sqrt{2} + i \cdot \frac{\pi}{4}$$

$$= \frac{1}{2}\log 2 + i\left(2n\pi + \frac{\pi}{4}\right)$$

13.8.6. Example. Resolve log cos (x + iy) into its real and imaginary parts.

Solution. cos (x + iy) = cos x cosh y – i sin x sinh y

= $\alpha + i\beta$, where α = cos x cosh y

and β = –sin x sinh y.

Now $\alpha^2 + \beta^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$$\frac{1+\cos 2x}{2}\cdot\frac{1+\cosh 2y}{2} + \frac{1-\cos 2x}{2}\cdot\frac{\cosh 2y - 1}{2} = \frac{1}{2}(\cos 2x + \cosh 2y),$$

and $$\frac{\beta}{\alpha} = -\tan x \tanh y$$

∴ Log cos (x + iy) = Log ($\alpha + i\beta$)

$$= 2n\pi i + \log\sqrt{\alpha^2+\beta^2} + i\tan^{-1}\frac{\beta}{\alpha}$$

$$= 2n\pi i + \frac{1}{2}\log\frac{\cos 2x + \cosh 2y}{2} - i\tan^{-1}(\tan x \tanh y)$$

Note. The method is that we write cos (x + iy) in the form α, β and then use formula for Log ($\alpha + \beta$).

These equations give $r = 1$ and $0 = \frac{\pi}{2}$.

∴ $\log(-i) = \log r + \theta i = \log 1 - \frac{\pi}{2}i = -\frac{\pi}{2}i$.

Note : Here we had to find the principal value of the logarithm of $-i$.

13.9. EXAMINATION ORIENTED EXERCISE

Evaluate

1. Log (–3)

2. Log i

3. Log (–5)

Resolve into real and imaginary parts.

4. Log sin $(x + iv)$

5. log cos $(x + iv)$

6. log (–1)

Prove that

7. $\log(x + iy) = \log(x^2 + y^2) + i \tan^{-1}\frac{y}{x}$.

8. $\log\frac{a+ib}{a-ib} = 2i\tan^{-1}\frac{b}{a}$

13.10. THE GENERAL EXPONENTIAL FUNCTION

We know that when a and x are real $a^x = e^{x\log a}$

We take this as the definition of the general exponential function a^z. when a and z are complex.

Thus, if a and z are complex

$$a^z = e^{z \log a}$$

Now, log a is many valued and so a^z is also many-valued.

We have $a^z = e^{z\log a} = e^z = e^{z(2n\pi + i \log a)}$

The value of a^2 obtained by putting n equal to zero is called its principal value.

13.10.1. The general logarithmic function.

Suppose a and z are complex.

$a^2 = w$, then z is called a logarithm of w to the base a and we write,

$z = \text{Log}_a w$.

13.10.2. Base-changing formula.

Let $a^2 = w$

Then, $e^{log a} = w$

$\therefore$ z Log $a = \text{Log}_e w$

But $z = \text{Log}_e w$

$\therefore$ $\text{Log}_a w. \text{Log}_e a = \text{Log}_e w$

or, $\text{Log}_a w. = \text{Log}_e w/\text{Log}\ _e a$.

13.10.3. Example. Separate $(\alpha + i\beta)^{x+iy}$ into real and imaginary parts.

Solution. $(\alpha + i\beta)^{x+iy} = e^{(x + iy)}\ \text{Log}^{(\alpha + \beta)}$

$= e^{(x + iy)\ (2n\pi i + \text{Log}\ r + i\theta)}$

where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1} \dfrac{\beta}{\alpha}$

$e^{\{x \log r - y(\theta + 2\pi x)\}} + i \{y \log r + x (\theta + 2n\pi)\}$

$= e^{u+iv}$, where $u = x \log r - y(\theta + 2n\pi)$

and $\quad v = y \log r - x(\theta + 2n\pi)$

$= e^{u}.e^{i\theta}$

$= e^{u}(\cos v + i \sin v)$

13.11. EXAMINATION ORIENTED EXERCISE

Prove that

1. $i^{a} = \cos (4m + 1) \dfrac{\pi a}{2} + i \sin(4m+1)\dfrac{na}{2}$.

2. If $\dfrac{(1+i)^{p+qi}}{(1-i)^{p-qi}} = \alpha + i\beta$ then one value of $\tan^{-1}\dfrac{\beta}{\alpha}$ is $p\pi + q \log 2$.

3. If i^{i} ....to $= \alpha + i\beta$, principal values only being considered, then

$$\tan \frac{\pi A}{2} = \frac{B}{A} \text{ and } A^2 + B^2 = e^{-\pi B}$$

4. If $\alpha^{a+i\beta} + (x + iy)^{p+qi}$, principal values only being considered, then

$$x = \frac{1}{2} p \log_0 (x^2 + y^2) - q \tan^{-1}\frac{y}{x} \log_a e.$$

and $\log (x^2 + y^2) = 2\dfrac{\alpha p + \beta q}{p^2 + q^2}$.

5. Prove that the principal value of $(a + ib)^{\alpha+\beta}$ is wholly real or wholly imaginary according as $\beta \log (a^2 + b^2) + \alpha \tan^{-1}\dfrac{b}{a}$ is an even or an odd multiple of $\dfrac{\pi}{2}$.

13.12. SUGGESTED READING

The students are advised to go through following references for details.

13.13. REFERENCES

(1) Functions of a Complex Variables by Goyal and Gupta, Pragati Prakashan, Meerut.

(2) Titu Andreescu and Dorin Andrica, *Complex Numbers from A to Z,* Birkhauser, 2006.

(3) A text Book of Real and Complex Analysis by Sunil Gupta, Udhay Banu, Ashok Kumar, Narinder Sharma, Malhotra Brothers, Pacca Danga, Jammu.

(4) James Ward Brown and Ruel V. Churchill, *Complex Variables and Applications,* 8th Ed., McGraw – Hill International Edition, 2009.

13.14. MODEL TEST PAPER

Separate into real and imaginary parts :

1. $\cos(\alpha + i\beta)$

2. $\cot(\alpha + i\beta)$

3. $\sec(\alpha + i\beta)$

4. $\text{cosec}(\alpha + i\beta)$

5. $\sinh(\alpha + i\beta)$

6. $\sinh\beta \sin\alpha + i\cosh\beta\cos\alpha = i\cos(a + i\beta)$

7. $\sin 2\alpha + i\sinh 2\beta = 2\sin(a + i\beta)\cos(a - i\beta)$

8. $\cos(\alpha + i\beta) + i\sin(\alpha + i\beta) = e^{-}\beta(\cos\alpha + i\sin\beta)$

9. If $\sin(A + B) = x + iy$, then $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1$

10. If $x + iy \cosh(u + iv)$, then $\dfrac{x^2}{\cos^2 v} - \dfrac{y^2}{\sin^2 v} = 1$ and $\dfrac{x^2}{\cosh^2 u} + \dfrac{y^2}{\sinh^2 u} = 1$.

11. Evaluate $\log(-1)$.

12. Prove that $\log(-i) = -\dfrac{\pi}{2}i$

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SUMMATION OF SERIES

14.1. Introduction : In this lesson the concept of summation of n terms of trigonometric series is discussed.

14.2 Objectives : Objective of studying this lesson is to explain the summation of n terms of trigonometric series.

14.3. To find the sum of a series of sines or consines of angles in A.P.

Let us find the sum to n terms of the following series

$\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots\ldots\ldots\ldots$

The angles are in A.P. their common differences being β.

Multiplying each term by $2\sin\frac{\beta}{2}$ we have

$$2\sin\alpha\sin\frac{\beta}{2} = \cos\left(\alpha - \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{\beta}{2}\right)$$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + \frac{3\beta}{2}\right)$$

$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{3\beta}{2}\right) - \cos\left(\alpha + \frac{5\beta}{2}\right)$$

....

....

$$2\sin(\alpha + \overline{n-1}\,\beta)\sin\frac{\beta}{2} = \cos\left(a + \overline{2n-3}\,\frac{\beta}{2}\right) - \cos\left(a + \overline{2n-1}\,\frac{\beta}{2}\right)$$

If S denote the sum to n terms, we have by addition

$$2\sin\frac{\beta}{2}.S = \cos\left(a - \frac{\beta}{2}\right) - \cos\left(a + \overline{2n-1}\frac{\beta}{2}\right)$$

$$= 2\sin\left(a + \overline{n-1}\,\frac{\beta}{2}\right)\sin\frac{n\beta}{2}$$

$$\therefore \qquad S = \frac{\sin\left(a + \overline{n-1}\,\frac{\beta}{2}\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

Let us now find the sum of the series

$$\cos\alpha + \cos(\alpha + \beta) + \text{..........to } n \text{ terms.}$$

Again multiplying each term by 2 sin $\frac{\beta}{2}$, we have

$$2\cos\alpha\sin\frac{\beta}{2} = \sin\left(\alpha + \frac{\beta}{2}\right) - \sin\left(\alpha - \frac{\beta}{2}\right)$$

$$2\cos(\alpha + \beta)\sin\frac{\beta}{2} = \sin\left(\alpha + \frac{3\beta}{2}\right) - \sin\left(\alpha + \frac{\beta}{2}\right)$$

$$2\cos(\alpha + 2\beta)\sin\frac{\beta}{2} = \sin\left(\alpha + \frac{5\beta}{2}\right) - \sin\left(\alpha + \frac{3\beta}{2}\right)$$

....

....

$$2\cos(\alpha + \overline{n-1}\,\beta)\sin\frac{\beta}{2} = \sin\left(\alpha + \overline{2n-1}\,\frac{\beta}{2}\right) - \sin\left(\alpha + \overline{2n-3}\frac{\beta}{2}\right)$$

If S denote the sum to n terms, we have by addition

$$2\sin\frac{\beta}{2}\cdot S = \sin\left(\alpha + \overline{2n-1}\,\frac{\beta}{2}\right) - \sin\left(\alpha - \frac{\beta}{2}\right)$$

$$= 2\cos\left(\alpha + n - 1\frac{\beta}{2}\right)\sin\frac{n\beta}{2}$$

$$\therefore \qquad S = \frac{\cos\left(\alpha + \overline{n-1}\frac{\beta}{2}\right)\sin\frac{n\,\beta}{2}}{\sin\frac{\beta}{2}}.$$

14.3.1. Example. Sum to n terms the series :

$$\sin x + \sin 2x + \sin 3x + ...$$

Solution. Here $\alpha = x$ and $\beta = x$

$$\therefore \qquad S = \frac{\sin\left(x + \overline{n-1}\frac{x}{2}\right)\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

$$= \frac{\sin(n+1)\frac{x}{2}\cdot\sin\frac{nx}{2}}{\sin\frac{x}{2}}$$

14.3.2. Example. Sum to n terms the series :

$$\cos\frac{\pi}{2n} + \cos\frac{3\pi}{2n} + \cos\frac{5\pi}{2n} + ...$$

Solution. Here $\alpha = \frac{\pi}{2n}$ and $\beta = \frac{\pi}{n}$

$$– \qquad S = \frac{\cos\left(\frac{\pi}{2n} + \overline{n-1}\frac{\pi}{2n}\right)\sin n\cdot\frac{\pi}{2n}}{\sin\frac{\pi}{2n}} = \frac{\cos\frac{\pi}{2}\cdot\sin\frac{\pi}{2}}{\sin\frac{\pi}{2n}} = 0$$

14.3.3. Example. Sum to n terms the series :

$$\cos^2 x + \cos^2 2x + \cos^2 3x + ...$$

Solution. $\cos^2 x = \dfrac{1+\cos 2x}{2}$

$$\cos^2 2x = \frac{1+\cos 4x}{2}$$

$$\cos^2 3x = \frac{1+\cos 6x}{2}$$

....

....

$\therefore \quad S = \dfrac{1}{2}(1+\cos 2x) + \dfrac{1}{2}(1+\cos 4x) + \dfrac{1}{2}(1+\cos 6x) + \text{.......to } n \text{ terms.}$

$$= \frac{n}{2} + \frac{1}{2}(\cos 2x + \cos 4x + \cos 6x + \text{........to } n \text{ terms})$$

$$= \frac{n}{2} + \frac{1}{2} \cdot \frac{\cos(2x + \overline{n-1}\,x)\sin nx}{\sin x}$$

$$= \frac{n}{2} + \frac{\cos(n+1)x \sin nx}{2\sin x}.$$

14.4. EXAMINATION ORIENTED EXERCISES

Sum up the following series upto n terms :

1. $\sin x + \sin (x - y) + \sin (x - 2y) + \ldots\ldots\ldots$
2. $\cos x + \cos 2x + \cos 3x + \ldots\ldots\ldots$
3. $\sin \alpha - \sin (\alpha + \beta) + \sin (\alpha + 2\beta) - \sin (\alpha + 3\beta) + \ldots\ldots\ldots$
4. $\cos x - \cos (x + y) + \cos (x + 2y) - \cos (x + 3y) + \ldots\ldots\ldots$.
5. $\sin x \cos x + \sin 2x \cos 2x + \sin 3x \cos 3x + \ldots\ldots\ldots\ldots$
6. $\cos^2\alpha - \cos^2(\alpha + \beta) + \cos^2 (\alpha + 2\beta) - \cos^2 (\alpha + 3\beta) + \ldots\ldots\ldots$
7. $\sin^2 x + \sin^2 2x + \sin^2 3x + \ldots\ldots\ldots\ldots\ldots$.
8. $\cos^3\alpha + \cos^3 3\alpha + \cos^3 5\alpha + \ldots\ldots\ldots$.
9. $\sin^3\alpha - \sin^3(\alpha + \beta) + \sin^3(\alpha + 2\beta) + \ldots\ldots\ldots\ldots$

10. $\sin\alpha \sin 2\alpha + \sin 2\alpha \sin 3\alpha + \sin 4\alpha + \ldots\ldots\ldots\ldots$.

11. Prove $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$

12. $\sinh u + \sinh (u + v) + \sinh (u + 2v) + \ldots\ldots\ldots\ldots$

13. $\cosh a + \cosh (a + b) + \cosh (a + 2b) + \ldots\ldots\ldots\ldots$.

14.5. METHOD OF DIFFERENCE

14.5.1. Formulaes

$\operatorname{cosec}\alpha = \cot\frac{\alpha}{2} - \cot\alpha$

$\tan\alpha = \cot\alpha - 2\cot 2\alpha.$

$\tan\alpha \sec 2\alpha = \tan 2\alpha - \tan\alpha$

$\operatorname{cosec}\alpha \operatorname{cosec}(\alpha + \beta) = \operatorname{cosec}\beta\,[\cot\alpha - \cot(\alpha + \beta)]$

$\sec\alpha \sec(\alpha + \beta) = \operatorname{cosec}\beta\,[\tan(\alpha + \beta) - \tan\alpha]$

$\tan^2\alpha \tan 2\alpha = \tan 2\alpha - 2\tan\alpha$

$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$

$\cos^3\alpha = \frac{1}{4}(3\cos\alpha + \cos 3\alpha)$

$\tan\alpha \tan(\alpha + \beta) = \cot\beta\,[\tan(\alpha + \beta) - \tan\alpha] - 1$

14.5.2. Example. Sum the series

$$\operatorname{cosec}\alpha + \operatorname{cosec} 2\alpha + \operatorname{cosec} 4\alpha + \ldots\ldots\ldots + \operatorname{cosec} 2^{n-1}\alpha.$$

Solution. $\operatorname{cosec}\alpha + \cot\alpha = \frac{1}{\sin\alpha} + \frac{\cos\alpha}{\sin\alpha}$

$$= \frac{1 + \cos\alpha}{\sin\alpha}$$

$$= \frac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$= \cot\frac{\alpha}{2}$$

$\therefore$ $$\operatorname{cosec} \alpha = \cot\frac{\alpha}{2} - \cot\alpha$$

$$\operatorname{cosec} 2\alpha = \cot \alpha - \cot 2\alpha$$

$$\operatorname{cosec} 4\alpha = \cot 2\alpha - \cot 2^2\alpha$$

....

$$\operatorname{cosec} 2^{n-1}\alpha = \operatorname{cosec} 2^{n-2}\alpha - \operatorname{cosec} 2^{n-1}\alpha$$

$\therefore$ Adding up, we get

$$S = \cot\frac{\alpha}{2} - \cot 2^{n-1}\alpha$$

14.5.3. Example. Sum up to n terms the series :

$$\tan \alpha + 2 \tan 2\alpha + 2^2 \tan 2^2\alpha + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution. $$\tan \alpha - \cot \alpha = \frac{\sin\alpha}{\cos\alpha} - \frac{\cos\alpha}{\sin\alpha}$$

$$= \frac{\sin^2\alpha - \cos^2\alpha}{\sin\alpha\cos\alpha}$$

$$= \frac{-\cos 2\alpha}{\frac{1}{2}\sin 2\alpha}$$

$\therefore$ $$\tan \alpha = \cot \alpha - 2 \cot 2\alpha$$

$$\tan 2\alpha = \cot 2\alpha - 2 \cot 2^2\alpha$$

$$\tan 2^2\alpha = \cot 2^2\alpha - 2 \cot 2^3\alpha$$

....

$$\tan 2^{n-1}\alpha = \cot 2^{n-1}\alpha - \cot 2^n\alpha$$

Multiplying by 1, 2, 2^2, ...2^{n-1} successively and adding we get

$$S = \cot \alpha - 2^n \cot 2^n \alpha.$$

14.5.4. Example. Sum the series :

$$\tan^{-1}\frac{1}{1+1+1^2}+\tan^{-1}\frac{1}{1+2+2^2}+\tan^{-1}\frac{1}{1+3+3^2}+\ldots\text{to } n \text{ terms}$$

Solution. Now $T_1 = \tan^{-1}\dfrac{1}{1+1+1^2} = \tan^{-1}\dfrac{1}{1+2} = \tan^{-1}\dfrac{2-1}{1+2.1}$

$\Rightarrow \qquad T_1 = \tan^{-1} 2 - \tan^{-1} 1 \qquad \left[\because \tan^{-1}\dfrac{x-y}{1+xy} = \tan^{-1}x - \tan^{-1}y\right]$

Also $\qquad T_2 = \tan^{-1}\dfrac{1}{1+2+2^2} = \tan^{-1}\dfrac{1}{1+6} = \tan^{-1}\dfrac{3-2}{1+3.2}$

$\Rightarrow \qquad T_2 = \tan^{-1} 3 - \tan^{-1} 2$

Similarly $\qquad T_3 = \tan^{-1} 4 - \tan^{-1} 3$

....

$T_n = \tan^{-1}(n+1) - \tan^{-1} n$

Adding vertically and cancelling like terms, we get

$$S_n = T_1 + T_2 + T_3 + \ldots\ldots\ldots\ldots + T_n = \tan^{-1}(n+1) - \tan^{-1} 1$$

$$= \tan^{-1}\frac{(n+1)-1}{1+(n+1)\cdot 1} = \tan^{-1}\left(\frac{n}{n+2}\right).$$

14.5.5. Example. $\dfrac{1}{\cos\theta+\cos 3\theta}+\dfrac{1}{\cos\theta+\cos 5\theta}+\dfrac{1}{\cos\theta+\cos 7\theta}+\ldots.$

Solution. Here $T_1 = \dfrac{1}{\cos\theta+\cos 3\theta}$

$$= \frac{1}{2\cos 2\theta\cos\theta} = \frac{1}{2\sin\theta}\left[\frac{\sin\theta}{\cos 2\theta\cos\theta}\right]$$

$$= \frac{\operatorname{cosec}\theta}{2}\left[\frac{\sin(2\theta-\theta)}{\cos 2\theta\cos\theta}\right]$$

$$= \frac{\operatorname{cosec}\theta}{2}\left[\frac{\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}{\cos 2\theta\cos\theta}\right]$$

$$= \frac{\operatorname{cosec}\theta}{2}\left[\frac{\sin 2\theta\cos\theta}{\cos 2\theta\cos\theta} - \frac{\cos 2\theta\sin\theta}{\cos 2\theta\cos\theta}\right]$$

$$= \frac{\operatorname{cosec}\theta}{2}\left[\tan 2\theta - \tan\theta\right]$$

$$\therefore \qquad T_1 = \frac{1}{\cos\theta + \cos 3\theta} = \frac{\operatorname{cosec}\theta}{2}\left[\tan 2\theta - \tan\theta\right]$$

Similarly $T_2 = \dfrac{1}{\cos\theta + \cos 5\theta} = \dfrac{\operatorname{cosec}\theta}{2}\left[\tan 3\theta - \tan 2\theta\right]$

$$T_3 = \frac{1}{\cos\theta + \cos 7\theta} = \frac{\operatorname{cosec}\theta}{2}\left[\tan 4\theta - \tan 3\theta\right]$$

....

$$T_n = \frac{1}{\cos\theta + \cos(2n+1)\theta} = \frac{\operatorname{cosec}\theta}{2}\left[\tan(n+1)\theta - \tan n\theta\right]$$

[nth term of 3, 5, 7, ...= 3 + (n – 1) × 2 = 2n + 1]

Adding vertically, we get the required sum

$$= \frac{\operatorname{cosec}\theta}{2}\left[\tan(n+1)\theta - \tan\theta\right]$$

$$= \frac{1}{2}\operatorname{cosec}\theta\left[\tan(n+1)\theta - \tan\theta\right].$$

14.6. EXAMINATION ORIENTED EXERCISES

Sum the following series to n terms :

1. $\operatorname{cosec}\alpha + \operatorname{cosec}\dfrac{\alpha}{2} + \operatorname{cosec}\dfrac{\alpha}{2^2} + ...$

2. cosec α cosec 2α + cosec 2α cosec 3α + cosec 3α cosec 4α + ...

3. $\tan\alpha\sec 2\alpha + \tan 2\alpha \sec 4\alpha + \tan 4\alpha \sec 8\alpha + ...$

4. $\sin^3\frac{\theta}{3} + 3\sin^3\frac{\theta}{3^2} + 3^2\sin^3\frac{\theta}{3^3} + ...$

5. $\frac{\sin\theta}{\sin 2\theta \sin 3\theta} + \frac{\sin\theta}{\sin 3\theta \sin 4\theta} + \frac{\sin\theta}{\sin 4\theta \sin 5\theta} + ...$

6. $\frac{1}{\cos\theta + \cos 3\theta} + \frac{1}{\cos\theta + \cos 5\theta} + \frac{1}{\cos\theta + \cos 7\theta} + ...$

7. $\tan\alpha \tan 2\alpha + \tan 2\alpha \tan 4\alpha + \tan 3\alpha \tan 8\alpha + ...$

8. $\tan^{-1}\frac{1}{1+1+1^2} + \tan^{-1}\frac{1}{1+2+2^2} + ... + \tan^{-1}\frac{1}{1+n+n^2}$

9. $\sum_{k=1}^{n} \tan^{-1}\left(\frac{1}{3+3k+k^2}\right).$

10. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + ...$

11. $\tan^{-1}\frac{4}{1+3\cdot 4} + \tan^{-1}\frac{6}{1+8\cdot 9} + \tan^{-1}\frac{8}{1+15\cdot 16} + ...$

12. $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + \tan^{-1}\frac{4}{33} + ...$

13. $\tan x + \frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2}............$

14.7. C + iS METHOD

List of some standard series.

Following formulaes will help students to solve C + *i*S method.

1. $(1 - x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + ...$

2. $(1 - x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + ...$

3. $(1 + x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + ...$

4. $(1 + x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1}{2}\cdot\frac{3}{4}x^2 - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^3 + ...$

5. $(1 - x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1}{2}\cdot\frac{3}{4}x^2 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^3 + ...$

6. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

7. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - ....$

8. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

9. $\sin h\ x = \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

10. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ....$

11. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$

12. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

13. $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ....$

14. $\log(1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ...\right)$

15.

16. $\tan^{-1} x = x -$

The method of summation will be illustrated with the help of an example.

14.7.1. Example. Sum to n terms, and to infinity, the series

$$1 + c \cos \alpha + c^2 \cos 2\alpha + ... ,$$

where c is less than one numerically.

Solution. Let

$$C = 1 + c \cos \alpha + c^2 \cos 2\alpha + ... + c^{n-1} \cos (n-1)\alpha$$

and $$S = c \sin \alpha + c^2 \sin 2\alpha + ...+ c^{n-1} \sin (n-1)\alpha.$$

Then, $C + iS = 1 + ce^{\alpha} + c^2 e^{2i\alpha} + ...+c^{n-1} e^{(n-1)i\alpha}$

$=$.

Now, we separate the right-hand expression into real and imaginary parts.

$=$

Hence, by equating real and imaginary parts, we get

$C =$

As n ∞, c^n and $c^{n+1} \to 0$

$\therefore$ Sum to infinity = $\dfrac{1 - c\cos\alpha}{1 - 2c\cos\alpha + c^2}$

Note. It may be noted that there are main steps in the process.

1. Forming C + i S.

2. Find the sum of the resulting G.P.

3. Resolving the sum into real and imaginary

14.7.2. Example. Sum the series to infinity.

$$S = \frac{1}{2}\sin\alpha + \frac{1\cdot 3}{2\cdot 4}\sin 2\alpha + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin 3\alpha + ...$$

Solution. Let $$C = \frac{1}{2}\cos\alpha + \frac{1\cdot 3}{2\cdot 4}\cos 2\alpha + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 3\alpha + ...$$

Then, $$C + iS = \frac{1}{2}e^{i\alpha} + \frac{1\cdot 3}{2\cdot 4}e^{2i\alpha} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}e^{3i\alpha} + ...\text{to infinite}$$

$$= (1 - e^{i\alpha})^{-\frac{1}{2}} \text{ by the Binomial theorem}$$

$$= \left[1 - (\cos\alpha + i\sin\alpha)\right]^{-\frac{1}{2}}$$

$$= \left(1 - \cos\alpha - i\sin\alpha\right)^{-\frac{1}{2}}$$

$$= \left(2\sin^2\frac{\alpha}{2} - 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)^{-\frac{1}{2}}$$

$$= \left(2\sin\frac{\alpha}{2}\right)^{-\frac{1}{2}}\left[\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) - i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right]^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2\sin\frac{\alpha}{2}}}\left[\cos\frac{\pi - \alpha}{4} + i\sin\frac{\pi - \alpha}{4}\right]$$

$\therefore$ Equating imaginary parts, we get

$$S = \frac{1}{\sqrt{2\sin\frac{\alpha}{2}}}.\sin\frac{\pi-\alpha}{4}$$

14.7.3. Example. Sum to infinity the series :

$$C = \cos\alpha - \frac{1}{3!}\cos(\alpha+2\beta) + \frac{1}{5!}\cos(\alpha+4\beta) - ......\text{to}\,\infty$$

Solution. Let $\quad S = \sin\alpha - \frac{1}{3!}\sin(\alpha+2\beta) + \frac{1}{5!}\sin(\alpha+4\beta) - ....\text{to}\,\infty$

Then, $\quad C + iS = e^{i\alpha} - \frac{1}{3!}e^{i(\alpha+2\beta)} + \frac{1}{5!}e^{i(\alpha+4\beta)} + ...\text{to}\,\infty$

$$= \frac{e^{i\alpha}}{e^{i\beta}}\left[e^{i\beta} - \frac{1}{3!}e^{3\beta} + \frac{1}{5!}e^{5i\beta} - ....\text{to}\,\infty\right]$$

$= e^{i(\alpha-\beta)}\cdot\sin e^{i\beta}$, using the series for sin z

$$= e^{i(\alpha-\beta)}\cdot\sin(\cos\beta + i\sin\beta)$$

$$= [\cos(\alpha-\beta) + i\sin(\alpha-\beta)]\left[\sin(\cos\beta)\cosh(\sin\beta) + i\cos(\cos\beta)\sinh(\sin\beta)\right]$$

$\therefore$ Equating real part, we gfet

$$C = \left[\cos(\alpha-\beta)\sin(\cos\beta)\cosh(\sin\beta) - \sin(\alpha-\beta)\cos(\cos\beta)\sinh(\sin\beta)\right]$$

14.7.4. Example. Find the sum to infinity of the series :

$$1 - \frac{1}{2}\cos\theta + \frac{1\cdot3}{2\cdot4}\cos2\theta - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cos3\theta + ...$$

Solution. Let $C = 1 - \frac{1}{2}\cos\theta + \frac{1\cdot3}{2\cdot4}\cos2\theta - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cos3\theta + ...\infty$

$\therefore \qquad S = 0 - \frac{1}{2}\sin\theta + \frac{1\cdot3}{2\cdot4}\sin2\theta - \frac{1\cdot3\cdot5}{2\cdot4\cdot6}\sin3\theta + ...\infty$

$$C + iS = 1 - \frac{1}{2}e^{i\theta} + \frac{1\cdot 3}{2\cdot 4}e^{2i\theta} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}e^{i3\theta} + \ldots.\infty$$

$$= 1 + \left(-\frac{1}{2}\right)e^{i\theta} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{\lfloor 2}e^{2i\theta} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{\lfloor 3}e^{3i\theta} + \ldots\infty$$

$$= (1+e^{i\theta})^{-1/2} = (1+\cos\theta + i\sin\theta)^{-1/2}$$

$$= \left(2\cos^2\frac{\theta}{2} + 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-1/2} = \left(2\cos\frac{\theta}{2}\right)^{-1/2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{-1/2}$$

$$= \left(2\cos^2\frac{\theta}{2}\right)^{-\frac{1}{2}}\left(\cos\frac{\theta}{4} - i\sin\frac{\theta}{4}\right)$$

Equating real parts.

14.7.5. Example. Sum the series :

$$1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha}{\lfloor 2}\cos 2\beta - \frac{\cos^3\alpha}{\lfloor 3}\cos 3\beta + \ldots\infty$$

Solution. Let $C = 1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha}{\lfloor 2}\cos 2\beta - \frac{\cos^3\alpha}{\lfloor 3}\cos 3\beta + \ldots\infty$

$$S = 0 - \cos\alpha\sin\beta + \frac{\cos^2\alpha}{\lfloor 2}\sin^2\beta - \frac{\cos^3\alpha}{\lfloor 3}\sin 3\beta + \ldots\infty$$

$$\therefore \quad C + iS = 1 - \cos\alpha(\cos\beta + i\sin\beta) + \frac{\cos^2\alpha}{\lfloor 2}(\cos 2\beta + i\sin 2\beta)$$

$$- \frac{\cos^3\alpha}{\lfloor 3}(\cos 3\beta + i\sin 3\beta) + \ldots.\infty$$

$$= 1 - \cos\alpha\cdot e^{i\beta} + \frac{\cos^2\alpha}{\lfloor 2}e^{2i\beta} - \frac{\cos^3\alpha}{\lfloor 3}e^{i3\beta} + \ldots.\infty$$

$$= 1 - z + \frac{z^2}{\underline{|2}} - \frac{z^3}{\underline{|3}} + ...\infty, z = e^{-i\beta}\cos\alpha$$

$$= e^{-z} = e^{-\cos\alpha}.e^{i\beta} = e^{-\cos\alpha(\cos\beta + i\sin\beta)}$$

$$= e^{-\cos\alpha\cos\beta}.e^{-i\cos\alpha\sin\beta}$$

$$= e^{-\cos\alpha\cos\beta}\left[\cos(\cos\alpha\sin\beta) - i\sin(\cos\alpha\sin\beta)\right]$$

Equating real parts,

$$C = e^{-\cos\alpha\cos\beta}.\cos(\cos\alpha\sin\beta).$$

14.8. EXAMINATION ORIENTED EXERCISES

Sum the series

1. $\sin\alpha + \frac{1}{2}\sin 2\alpha + \frac{1}{2^2}\sin 3\alpha + ...$ to infinity.

2. $\sin\alpha + c\sin(\alpha+\beta) + c^2\sin(\alpha+2\beta) + ...$ to n terms & to ∞.

3. $\sin\alpha\ \sec\alpha + \sin 2\alpha\ \sec^2\alpha + \sin 3\alpha\ \sec^3\alpha + ...$to n terms.

4. $\cos\theta\ \cos\theta + \cos^3\theta\ \cos 3\theta + \cos^5\theta\ \cos\theta + ...$to n terms.

5. $\cos\theta + \frac{1}{3}\cos 2\theta + \frac{1}{3^2}\cos 3\theta + ...$to n terms.

6. $\sin\alpha + \frac{1}{2}\sin 3\alpha + \frac{1\cdot 3}{2\cdot 4}\sin 5\alpha + ...\text{to }\infty$

7. $1 - \cos\alpha\cos\beta + \frac{\cos^2\alpha}{2!}\cos 2\beta - \frac{\cos^3\alpha}{3!}\cos 3\beta + ...\text{to }\infty$

8. $\sin\alpha + x\sin(\alpha+\beta) + x^2\sin(\alpha+2\beta) + ...$to n terms

9. $1 + \frac{1}{2}\cos\alpha + \frac{1\cdot 3}{2\cdot 4}\cos 2\alpha + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos 3\alpha + ...\text{to}\infty$

10. $\cos\theta + \frac{\sin\theta\cos 2\theta}{1!} + \frac{\sin^2\theta\cos 3\theta}{2!}.......\text{to}\infty$

14.9. SUGGESTED READING

The students are advised to go through following references for details.

14.10. REFERENCES

(1) Functions of a Complex Variables by Goyal and Gupta, Pragati Prakashan, Meerut.

(2) Titu Andreescu and Dorin Andrica, *Complex Numbers from A to Z,* Birkhauser, 2006.

(3) A text Book of Real and Complex Analysis by Sunil Gupta, Udhay Banu, Ashok Kumar, Narinder Sharma, Malhotra Brothers, Pacca Danga, Jammu.

(4) James Ward Brown and Ruel V. Churchill, *Complex Variables and Applications,* 8th Ed., McGraw – Hill International Edition, 2009.

14.11. MODEL TEST PAPER

Q.1. Find the sum to infinity of the series

$$1 - \frac{1}{2}\cos\theta + \frac{1}{2}.\frac{3}{4}\cos 2\theta - \frac{1}{2}.\frac{3}{4}.\frac{5}{6}\cos 3\theta + ...$$

Q.2. Find the sum to infinity S = $x\sin\alpha + \frac{1}{3}x^3 \sin 3\alpha + \frac{1}{5}x^5 \sin 5\alpha + ...$

Q.3. Find the sum to infinity S = $\sin\alpha + x\sin(\alpha + \beta) + x^2 \frac{\sin(\alpha + 2\beta)}{2!} + ...$

Q.4. Find the sum to infinity S = $c\cos\alpha + \frac{c^2}{2}\cos 2\alpha + \frac{c^3}{3}\cos 3\alpha + ...$

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SELF LEARNING MATERIAL

B.A. SEMESTER-IV

SUBJECT : MATHEMATICS UNIT I **to V**

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