Directorate of Distance Education

UNIVERSITY OF JAMMU
JAMMU

STUDY MATERIAL
FOR
B.COM, SEMESTER - I

Course No. : BC 104 **Unit I to V**

Business Mathematics **Lesson No. 1 to 25**

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http:/www.distanceeducationju.in

Printed and Published on behalf of the Directorate of Distance Education, University of Jammu by the Director, DDE, University of Jammu, Jammu.

Course No. **BC-104** Title : **Business Mathematics**

Duration of Exam. : **3 Hrs.** Total marks : **100**

Theory Examination: **80**

Internal Assessment : **20**

OBJECTIVE : To impart knowledge about fundamental mathematics used in business.

UNIT – I : COMMERCIAL ARITHMETIC

Introduction of business mathematics; Scope and importance of quantitative techniques; Concept of equated monthly instalment (EMI), profit and loss, simple and compound interest including half yearly and quarterly calculations, bill of discounting-Business applications.

UNIT – II: SET THEORY

Concept of a set , operation of sets, Algebra of sets, Cartesian product of two sets and its application to business mathematics.

UNIT – III : PROGRESSIONS

Arithmetics progression, Finding the nth term, Sum of n terms, representation of an A.P. Geomatric progression , Finding the nth term , Sum of n terms and sum of infinity, representation of an G.P. Special cases $\sum \boldsymbol{n}$, $\sum \boldsymbol{n}^2$, $\sum \boldsymbol{n}^3$

UNIT – IV : MATRIX AND MEASURMENT

Concept of a matrix , algebra of matrices, inverse of matrices, determinant of a square matrix, expansion rule, properties of determinant, solution of a system of linear equation upto 3 variable using 1. Cramer's Rule 2. The method of matrix inverse.

SKILL DEVELOPMENT (SPECIMEN FOR CLASS ROOM TEACHING AND INTERNAL ASSESSMENT)

A minimum of five exercises to be undertaken from above said courses selecting atleast from one unit.

BOOKS RECOMMENDED:

1.	Dr. A.K. Arte & R.V. Prabhakar	: A Text book of Business-Mathematics
2.	Dorai Raj	: Business Mathematics
3.	Sanchethi & Kapoor	: Business Mathematics
4.	Zamiruddin & khanna	: Business Mathematics
5.	Saha	: Business Mathematics
6.	Kavita Gupta	: Business Mathematics
7.	VK Kapoor	: Linear Programming

NOTE FOR PAPER SETTER:

Equal weightage shall be given to all the units of the syllabus. The external paper shall be of the two sections viz, A&B.

Section-A: This section will contain four short answer questions selecting one from each unit. Each question carries 5 marks. A candidate is required to attempt all the four questions. Total weightage to this section shall be 20 marks.

Section- B: This section will contain eight long answer questions of 15 marks each. Two questions with internal choice will be set from each unit. A candidate has to attempt any four questions selecting one from each unit. Total weightage to this section shall be 60 marks

MODEL QUESTION PAPER

BUSINESS MATHEMATICS

Section – A (20 Marks)

Attempt all questions. Each question carries five marks.

1. Explain the importance of quantative techniques in business?
2. Explain the concept of set with suitable examples?
3. Differentiate between arithmetic and geometric progression?
4. Explain the concept of expansion rule?

Section – B (60 Marks)

Attempt any four questions selecting one question from each unit. Each question carries 15 marks.

1. A sum of Rs. 800 amounts to Rs. 920 in 3 years at a simple inerest. If the interest rate is increased by 3% , it would amount to how much.

   OR

   What annual instalment will discharge a debt of Rs.1092 due in 3 years at 125 simple interest.

2. Which of the following sets are equal.

   A={0,L,W,F}

   B={Letters of the word follow}

   C={Letters of the word wolf}

   D={Letters of the word flow}

   OR

   Find true & False for the following set

   V={1,2,3,4,5}

   A={1,2,3}

B={3,5}

C={2,4}

3. What is arithmetic progression. Explain with suitable example?

OR

What is the Geographic progression. Explain with example ?

4. Evaluate

a. $\begin{vmatrix} 5 & 4 \\ -2 & 3 \end{vmatrix}$ b. $\begin{vmatrix} x^{-1} & 1 \\ X^3 & X^2+X+1 \end{vmatrix}$ c. $\begin{vmatrix} X^2+xy+y^2 & X+Y \\ X^2-xy+y^2 & X-Y \end{vmatrix}$

OR

Use matrix method to solve the following system of equations 5x-&y=2, 7x-5y=3

STRUCTURE

1.1 INTRODUCTION

Business mathematics is mathematics used by commercial enterprises to record and manage business operations. Commercial organisations use mathematics in accounting, inventory management, marketing, sales forecasting, and financial analysis. Mathematics typically used in commerce includes elementary arithmetic, elementary algebra, statistics and probability. Business management an be done more effective in some cases by use of more advanced mathematics such as calculus, matrix algebra and linear programming.

Business mathematics, sometimes called commercial math or consumer math, is a group of practical subjects used in commerce and everyday life. In schools, these subjects are often taught to students who are not planning a university education. In the United States, they are typically offered in high schools and in schools that grant associate's degrees.

The emphasis in these courses is on computational skills and their practical application, with practical application being predominant. A U.S. business math course might include a review of elementary arithmetic, including fractions, decimals, and percentages.

Elementary algebra is often included as well, in the context of solving practical business problems. The practical applications typically include checking accounts, price discounts payroll calculations, simple and compound interest, consumer and business credit, and mortgages and [revenues].

Business mathematics is mathematics used by commercial enterprises to record and manage business operations. Commercial organisations use mathematics in accounting, inventory management, marketing, sales forecasting, and financial analysis. Mathematics typically used in commerce includes elementary arithmetic, elementary algebra, statistics and probability. Business management can be made more effective in some cases by use of more advanced mathematics such as calculus, matrix algebra and linear programming. Business Mathematics is very important for modern business management. The forecasting and operating procedures are based primarily on business mathematics. Things such as simple interest and compound interest show a company what it will lose or get over the years if it invests in a particular asset. Business mathematics also helps in cost and price calculations which are the basis of cash inflows and outflows that all companies have to deal with. In academia, Business Mathematics includes mathematics courses taken at an undergraduate level by business students at University. These courses are slightly less difficult and do not always go into the same depth as other mathematics courses for people majoring in mathematics or science fields. The two most common math courses taken in this form are Business Calculus and

Business Statistics. Examples used for problems in these courses are usually real-life problems from the business world to help students gain a more detailed understanding.

Thus, to conclude Business Mathematics in management system is more effective in some cases by the use of more advanced mathematics such as calculus, matrix algebra and linear programming. Business organisations use mathematics in accounting, inventory management, marketing, sales forecasting, financial analysis etc.

1.2 OBJECTIVE:

After reading this unit you would be able:

- To understand importance of quantitative techniques.
- To know about applications of bills of discounting in business.
- To provide the student clear concept about profit and loss.
- To enhance the student in purchasing and selling activity.
- To make recognise the student with money value in marketing system.

1.3 SCOPE OF QUANTITATIVE TECHNIQUES

Quantitative techniques may be defined as those techniques which provide the decision maker a systematic and powerful means of analysis, based on quantitative data. It is a scientific method employed for problem solving and decision making by the management. With the help of quantitative techniques, the decision maker is able to explore policies for attaining the pre-determined objectives. In short, quantitative techniques are inevitable in decision-making process.

There are different types of quantitative techniques. We can classify them into three categories. They are: 1. Mathematical Quantitative Techniques 2. Statistical Quantitative Techniques 3. Programming Quantitative Techniques.

1.3.1 Mathematical Quantitative Techniques:

A technique in which quantitative data are used along with the principles of mathematics is known as mathematical quantitative techniques. Mathematical quantitative techniques involve: 1. Permutations and Combinations: Permutation means arrangement of objects in a definite order. The number of arrangements depends upon the total number of objects and the number of objects taken at a time for arrangement. Combination means selection or grouping objects without considering their order. 2. Set Theory: - Set theory is a modern mathematical device which solves various types of critical problems. Matrix Algebra: Matrix is an orderly arrangement of certain given numbers or symbols in rows and columns. It is a mathematical device of finding out the results of different types of algebraic operations on the basis of the relevant matrices. 4. Determinants: It is a powerful device developed over the matrix algebra. This device is used for finding out values of different variables connected with a number of simultaneous equations. 5. Differentiation: It is a mathematical process of finding out changes in the dependent variable with reference to a small change in the independent variable. 6. Integration: Integration is the reverse process of differentiation. 7. Differential Equation: It is a mathematical equation which involves the differential coefficients of the dependent variables.

1.3.2 Statistical Quantitative Techniques:

Statistical techniques are those techniques which are used in conducting the statistical enquiry concerning to certain Phenomenon. They include all the statistical methods beginning from the collection of data till interpretation of those collected data. Statistical techniques involve: 1. Collection of data: One of the important statistical methods is collection of data. There are different methods for collecting primary and secondary data. 2. Measures of Central tendency, dispersion, skewness and Kurtosis Measures of Central tendency is a method used for finding he average of a series while measures of dispersion used for finding out the variability in a series. Measures of Skewness measures asymmetry of a distribution while measures of Kurtosis measures the flatness of peakedness in a distribution. 3. Correlation and Regression Analysis: Correlation is used to study the degree of relationship among two or more variables. On the other hand, regression technique is used to estimate the value of one variable for a given value of another. 4. Index Numbers: Index numbers measure the fluctuations in various Phenomena like price, production etc over a period of time, they are described as economic barometres. 5. Time series Analysis: Analysis of time series helps us to know the effect of factors which are responsible for changes: 6. Interpolation and Extrapolation: Interpolation is the statistical technique of estimating under certain assumptions, the missing figures which may fall within the range of given figures. Extrapolation provides estimated figures outside the range of given data. 7. Statistical Quality Control Statistical quality control is used for ensuring the quality of items manufactured. The variations in quality because of assignable causes and chance causes can be known with the help of this tool. Different control charts are used in controlling the quality of products. 8. Ratio Analysis: Ratio analysis is used for analyzing financial statements of any business or industrial concerns which help to take appropriate decisions. 9. Probability Theory: Theory of probability provides numerical values of the likely hood of the occurrence of events. 10. Testing of Hypothesis: Testing of hypothesis is an important statistical tool to judge the reliability of inferences drawn on the basis of sample studies.

1.3.3 Programming Techniques: Programming techniques are also called operations research techniques. Programming techniques are model building techniques used by decision makers in modern times. Programming techniques involve: 1. Linear Programming: Linear programming technique is used in finding a solution for optimizing a given objective under certain constraints. 2. Queuing Theory: Queuing theory deals with mathematical study of queues. It aims at minimizing cost of both servicing and waiting. 3. Game Theory: Game theory is used to determine the optimum strategy in a competitive situation. 4. Decision Theory: This is concerned with making sound decisions under conditions of certainty, risk and uncertainty. 5. Inventory Theory: Inventory theory helps for optimizing the inventory levels. It focuses on minimizing cost associated with holding of inventories. 6. Net work programming: It is a technique of planning, scheduling, controlling, monitoring and co-ordinating large and complex projects comprising of a number of activities and events. It serves as an instrument in resource allocation and adjustment of time and cost up to the optimum level. It includes CPM, PERT etc. 7. Simulation: It is a technique of testing a model which resembles a real life situations 8. Replacement Theory: It is concerned with the problems of replacement of machines, etc due to their deteriorating efficiency or breakdown. It helps to determine the most economic replacement policy. 9. Non Linear Programming: It is a programming technique which involves finding an optimum solution to a problem in which some or all variables are non-linear. 10. Sequencing: Sequencing tool is used to determine a sequence in which given jobs should be performed by minimizing the total efforts. 11. Quadratic Programming: Quadratic programming technique is designed to solve certain problems, the objective function of which takes the form of a quadratic equation. 12. Branch and Bound Technique It is a recently developed technique. This is designed to solve the combinational problems of decision making where there are large number of feasible solutions. Problems of plant location, problems of determining minimum cost of production etc. are examples of combinational problems.

1.3.4 Objectives of Quantitative Techniques

1. To facilitate the decision-making process.

2. To provide tools for scientific research.

3. To help in choosing an optimal strategy.

4. To enable in proper deployment of resources.

5. To help in minimising costs.

6. To help in minimising the total processing time required for performing a set of jobs.

1.4 Importance of Quantitative Techniques

Quantitative techniques render valuable services in the field of business and industry. Today, all decisions in business and industry are made with the help of quantitative techniques. Some important uses of quantitative techniques in the field of business and industry are given below:

1. Quantitative techniques of linear programming is used for optimal allocation of scarce resources in the problem of determining product mix

2. Inventory control techniques are useful in dividing when and how much items are to be purchase so as to maintain a balance between the cost of holding and cost of ordering the inventory

3. Quantitative techniques of CPM and PERT helps in determining the earliest and the latest times for the events and activities of a project. This helps the management in proper deployment of resources.

4. Decision tree analysis and simulation technique help the management in taking the best possible course of action under the conditions of risks and uncertainty.

5. Queuing theory is used to minimise the cost of waiting and servicing of the customers in queues.

6. Replacement theory helps the management in determining the most economic replacement policy regarding replacement of equipment.

1.4.1 Applications of Quantitative Techniques in:

i) Marketing:

- Analysis of marketing research information
- Statistical records for building and maintaining an extensive market
- Sales forecasting

ii) Production

- Production planning, control and analysis
- Evaluation of machine performance

- Quality control requirements
- Inventory control measures

iii) Finance, Accounting and Investment:

- Financial forecast, budget preparation
- Financial investment decision
- Selection of securities
- Auditing function
- Credit, policies, credit risk and delinquent accounts

iv) Personnel:

- Labour turnover rate
- Employment trends
- Performance appraisal
- Wage rates and incentive plans

v) Economics

- Measurement of gross national product and input-output analysis
- Determination of business cycle, long-term growth and seasonal fluctuations
- Comparison of market prices, cost and profits of individual firms
- Analysis of population, land economics and economic geography
- Operational studies of public utilities
- Formulation of appropriate economic policies and evaluation of their effect

vi) Research and Development

- Development of new product lines
- Optimal use of resources
- Evaluation of existing products

1.4.2 The following are the important limitations of quantitative techniques:

Even though the quantitative techniques are inevitable in decision-making process, they are not free from short comings.

1. Quantitative techniques involves mathematical models, equations and other mathematical expressions
2. Quantitative techniques are based on number of assumptions. Therefore, due care must be ensured while using quantitative techniques, otherwise it will lead to wrong conclusions.
3. Quantitative techniques are very expensive.
4. Quantitative techniques do not take into consideration intangible facts like skill, attitude etc.
5. Quantitative techniques are only tools for analysis and decision-making. They are not decisions itself.

1.5 Equated Monthly Instalment (EMI)

1.5.1 Concept

A fixed payment amount made by a borrower to a lender at a specified date each calendar month. Equated monthly instalments are used to pay off both interest and principal each month, so that over a specified number of years, the loan is paid off in full. With most common types of loans, such as real estate mortgages, the borrower makes fixed periodic payments to the lender over the course of several years with the goal of repaying the loan. EMIs differ from variable payment plans, in which the borrower is able to pay higher payment amounts at his or her discretion. In EMI plans, borrowers are usually allowed one fixed payment amount each month

1.5.2 Calculation of EMI (Equated Monthly Installments)

When a person takes loan from a bank to buy a house for a fixed tenure, he has to pay EMI (*Equated Monthly Installments*). EMI is the term used for the monthly payment made by a borrower to the lender towards interest and principal money borrowed.

EMI amount depends on the following factors:

Amount of Loan: EMI depends primarily on the amount of loan you have taken. With an increase in the loan amount, the EMI to be paid also increases.

Tenure of the Loan: Next most important factor is the time for which you have taken the loan. The EMI decreases with the increase in the tenure of the loan. But one should understand the increase in tenure means that you will have to pay more interest to the bank. Since you will be having an outstanding amount against you for a longer time you will have to pay some extra for taking more time. EMI increases with the shorter tenure; in this case one should do the budgeting properly and make sure that the EMI can be paid on time.

Complete your Loan as Soon as Possible: For a longer tenure of loan one has to pay more interest, so one should plan prudently and pay back the loan as soon as possible.

Fluctuation in Interest Rates: Interest rate is a floating parameter. It keeps changing with Inflation and changing policies of the government. One has to pay the EMI according to the prevailing interest rates. RBI has recently cut repo rate by 50 bases. Many banks have already reduced the interest rates accordingly, in this case the buyer will have to pay the EMIs on the reduced rates and this will make a significant difference. Formula for finding EMI is given as under:

The mathematical formula to calculate EMI is:

EMI = P x r x (1 + r) n/ ((1 + r)n - 1)

Where P= Loan amount, r = interest rate, n=tenure in number of months.

Let's try to understand it using an example.

Suppose Ram has borrowed Rs. 5 lakhs from a bank on the interest rate of 12 per cent for 10 years.

Do in this case:

M (Loan period in months) : No of Years X 12 = 10 X 12 = 120 I (Interest rate per Annum / 12) : (12/100) / 12 = .01

L(LoanAmount : Rs. 5 ,00,000 EMI (Equated Monthly Instalments)

$$(5,00,000 \times .01) \times \frac{(1+.01)^{120}}{[(1+.01)^{120}]-1} = \text{Rs } 7174$$

So, in the example discussed above:

- EMI that Ram has to pay is Rs. 7147.
- Total payment made by Ram to the bank in 10 years (EMI X Total tenure in months (7174 X 120) is Rs 8, 60, 880.
- The total interest rate payable will be (Total payment – loan amount) Rs. 3,60,88

1.5.3 Reasons for Varied EMI Payments

The other major factor which determines the EMI payments is the type of interest on the loan. In case of fixed rate loans, the EMI payments remain constant during the tenure. In case of floating rate loans, the interest rates vary based on the prevailing market rates. Hence, the EMI payments also vary whenever there is a change in the base rates. The other factor which effects the EMI payments is the pre-closure or partial payments made towards the loan. Any partial payments made towards the loan are deducted from the principal amount of the loan. This results in reduction of total interest that is to be paid. Generally an individual who is making a partial payment will be given an option to keep the tenure constant or keep the EMI constant. If one opts for keeping the tenure constant, the monthly EMI payments will be reduced. Similarly, if one opts for keeping the EMI constant, the tenure of the loan will be reduced.

1.5.4 Floating Rate EMI Calculation

Calculate floating or variable rate EMI by taking into consideration two opposite scenarios, i.e., optimistic (deflationary) and pessimistic (inflationary) scenario. Loan amount and loan tenure are the two components required to calculate the EMI i.e., you are going to decide how much loan you have to borrow and how long your loan tenure should be. But interest rate is decided by the banks & HFCs based on rates and policies set by RBI. As a borrower, you should consider the two extreme possibilities of increase and decrease in the rate of interest and calculate how much would be your EMI under these two conditions. Such calculation will help you decide how much EMI is affordable, how long your loan tenure should be and how much you should borrow.

Optimistic (deflationary) scenario: Assume that the rate of interest comes down by 1% – 3% from the present rate. Consider this situation and calculate your EMI. In this situation, your EMI will come down or you may opt to shorten the loan tenure. Ex: If you avail home loan to purchase a house as an investment, then optimistic scenario enables you to compare this with other investment opportunities.

Pessimistic (inflationary) scenario: In the same way, assume that the rate of interest is hiked by 1% – 3%. Is it possible for you to continue to pay the EMI without much struggle? Even a 2% increase in rate of interest can result in significant rise in your monthly payment for the entire loan tenure. Such calculation helps you to plan for such future possibilities. When you take a loan, you are making a financial commitment for next few months, years or decades.

1.6 PROFIT AND LOSS

In our day to day life, we come across a number of situations wherein we use the concept of percent. Here, in this section we discuss the application of percentage in problems of profit and loss, & discount. Profit and loss is the branch of basic mathematics which deals with the study of profit and loss made in a business transaction. The profit and loss account is fundamentally a summary of the trading transactions of a business and shows whether it has made a profit or loss during a particular period of account. Indeed, by deducting the total expenditure from total income the profit or loss of a business can be calculated. Along with the balance sheet, it is one of the key financial statements that make up a company's statutory accounts.

Let us begin with the terms and formulae related to profit and loss.

Cost Price (C.P.): The Price at which an article is purchased, is called its cost price.

Selling Price (S.P.): The Price at which an article is sold, is called its selling price.

Profit (Gain): When S.P. > C.P., then there is profit, and

Profit = S.P. – C.P.

Loss: When C.P. > S.P., then there is loss, and Loss = C.P. – S.P.

Formulae

Profit % = (Profit/CP * 100) %

Loss% **=** (Loss/CP *100) %

One thing always kept in mind that Gain % or loss % is always calculated on C.P.

Example 1: A shopkeeper buys an article for Rs. 360 and sells it for Rs. 270. Find his
gain or loss percent.

Sol: Here C.P. = Rs. 360, and S.P. = Rs. 270

Since C.P. > S.P.,

∴ there is a loss.

Loss = C.P. – S.P. = Rs (360 – 270) = Rs. 90

Loss % = (loss/CP*100) %

= 25%

Example 2: Sudha purchased a house for Rs. 4, 52, 000 and spent Rs. 28,000 on its repairs. She had to sell it for Rs. 4, 92, 000. Find her gain or loss percent.

Sol: Here C.P. = Cost price + Overhead charges

= Rs. (452000 + 28000) = Rs. 4,80,000

S.P. = Rs. 4,92,000

Since, S.P. > C.P., ∴ Gain = Rs. (492000 – 480000) = Rs. 12000

Gain % = 2.5%

Example 3: By selling a book for Rs. 258, a publisher gains 20%. For how much should he sell it to gain 30%?

Sol: S.P. = Rs. 258

Profit = 20%

C.P. = SP*100/100+ profit%

= Rs. 215

Now, if Profit = 30% and C.P. = Rs. 215, then,

S.P. = CP (100+profit %) / 100

= 215×130/ 100

= Rs. 279.50

Example 4: A man bought oranges at 25 for Rs. 100 and sold them at 20 for 100. Find his gain or loss percent.

Sol: C.P. of 25 oranges = Rs. 100

∴ C.P. of 1 orange = Rs. 100/ 25

= Rs. 4

and S.P. of 1 orange = Rs. 100/ 20

= Rs. 5

∴ Profit on 1 orange = Rs (5 – 4) = Rs. 1

Profit % = 100 25%

Example 5: A man sold two horses for Rs. 29700 each. On one he lost 10% while he gained 10% on the other. Find his total gain or loss percent in the transaction.

Sol: S.P. of first horse = Rs. 29700

Loss = 10%

∴ C.P. = Rs. 29700×100/90

= Rs. 33,000

S.P. of 2nd horse = Rs. 29700,

Profit = 10%

C.P. = Rs. 29700×100/110

= Rs. 27,000

Total CP = Rs. (33000 + 27000) = Rs. 60,000

Total SP = Rs. (2 × 29700) = Rs. 59400

Net Loss = Rs. (60000 – 59400) = Rs. 600

Loss % = 600/60000*100

= 1%

Example 6: The cost price of 15 articles is equal to the selling price of 12 articles. Find the gain percent.

Sol: Let the C.P. of 15 articles be Rs. 15

then S.P. of 12 articles = Rs. 15

S.P. of 15 articles = Rs. 15/12*15

= 75/4

75

Gain = (75/4 – 15) = Rs. 15/4

Gain % = 15/4/15*100 = 25%

Example 7: A watch was sold at a profit of 12%. Had it been sold for 33 more, the profit would have been 14%. Find the cost price of the watch.

Sol: Let the cost price of the watch be ` *x*

Then S.P = x*112/100 = 112x/100.

If the watch is sold for Rs. 33 more then S.P. = (112x/100 +33)

New profit 14%

CP. = x= (112x/100+ 33)× 100/114

or 114x = 112 x + 3300 or 2x = 3300

x = 1650 ∴C.P. = ` 1650

1.7 SIMPLE INTEREST

Interest is money paid by a borrower to a lender for a credit or a similar liability. Important examples are bond yields, interest paid for bank loans, and returns on savings. Interest differs from profit in that it is paid to a lender, whereas profit is paid to an owner. In economics, the various forms of credit are also referred to as loanable funds. When money is borrowed, interest is typically calculated as a percentage of the principal, the amount owed to the lender. The percentage of the principal that is paid over a certain period of time (typically a year) is called the interest rate. Interest rates are market prices which are determined by supply and demand. They are generally positive because loanable funds are scarce. Interest is often compounded, which means that interest is earned on prior interest in addition to the principal. The total amount of debt grows exponentially, and its mathematical study led to the discovery of the number *e*. In practice, interest is most often calculated on a daily, monthly, or yearly basis, and its impact is influenced greatly by its compounding rate.

When a person has to borrow some money as a loan from his friends, relatives, bank etc. he promises to return it after a specified time period along with some extra money for using the money of the lender. The money borrowed is called the Principal, usually denoted by *P*, and the extra money paid is called the Interest, usually denoted by I.

The total money paid back, that is, the sum of Principal and the Interest is called the Amount, and is usually denoted by A.

Thus, A = P + I

The interest is mostly expressed as a rate percent per year (per annum).

Interest depends on, how much money (P) has been borrowed and the duraton of time (T)

for which it is used. Interest is calculated according to a mutually agreed rate percent, per annum (R). [i.e. R = r % = r/100]

Thus, Interest = (Principal) × (Rate % per annum) × time

∴ I = P × R × T

Interest calculated as above, is called simple interest.

Example 1: Find the simple interest in each of the following cases:

(a) P= Rs. 8000 R= 5% T= 2 yrs

(b) Rs. 20,000 R= 15% T= 1*1/2 years

Sol: (a) I= 8000* 5* 2/100 = Rs. 800

(b) I= 20000* 15* 3/2*100= Rs. 4500

Example 2: Find at what rate of simple interest per annum will 5000 amount to Rs. 6050 in 3 years.

Sol: Here A = Rs. 6050, P = Rs. 5000, T = 3 yrs
∴ I = Rs. (6050 – 5000) = Rs. 1050
I = P × R × T or r% = I/ P*T
R= 1050*100/5000*3 = 7%

Example 3: A sum amounts to Rs. 4875 at $12\times\frac{1}{2}\%$ simple interest per annum after 4
years. Find the sum.

Sol: Here A = Rs. 4875, R = 12-1/2 %, T= 4 years

I = P × R × T

I = Rs. P* 25/200* 4 = Rs. P/2

∴ A = P + P/2 =Rs. 3P/2
Thus, 3P/2 = 4875 => 3P= 9750=> P= Rs. 3250

Example 4: In how many years will a sum of Rs. 2000 yield an interest (Simple) of
Rs. 560 at the rate of 14% per annum?

Sol: Here P = Rs. 2000, I = Rs. 560 R = 14%
I = P × R × T or 560 = 2000*14/100*T
∴ T = 560 *100/ 2000* 14 = 2 years

Example 5: A certain sum of money at simple interest amounts to Rs. 1300 in 4 years
and to Rs. 1525 in 7 years. Find the sum and rate percent.

Sol: Amount after 4 years = Rs. 1300
Amount after 7 years = Rs. 1525
∴Interest for 3 years = Rs. [1525 – 1300] = Rs. 225

∴ Interest for 1 year = Rs. 225/3 = 75

∴ 1300 = P + Interest for 4 yrs = P + 4 × 75 or P = Rs. (1300 – 300) = Rs. 1000

R = 75 ×100/ 1000 ×1 = 7.5%

Example 6: A certain sum of money doubles itself in 10 years. In how many years will it become $2\times 1/2$ times at the same rate of simple interest.

Sol: Let P = Rs. 100, T = 10 yrs, A = Rs. 200, ∴ I = Rs. 100

∴100 = $100\times \frac{R \times 10}{100}$ or R = 10%

Now P = Rs. 100, R = 10% and A = Rs. 250 4 I = Rs. 150

∴150 = $100\times \frac{10 \times T}{100} = T = 15\,years$

Thus, in 15 yrs, the sum will become $2\frac{1}{2}$ *times*

Example 7: Out of ` 70,000 to invest for one year, a man invests Rs. 30,000 at 4% and Rs. 20,000 at 3% per annum simple interest. At what rate percent, should he lend the remaining money, so that he gets 5% interest on the total amount he has?

Sol: Interest on total amount at 5% for one year = Rs. 70, 000$\times \frac{5}{100} \times 1$

= Rs. 3500

Interest on Rs. 30,000 at 4% for 1 year = Rs. 30000 $\times \frac{4}{100} \times 1$

= Rs. 1200

Interest on ` 20,000 at 3% for 1 year = 20000 $\times \frac{3}{100} \times 1$

= Rs. 600

∴ Interest on remaining Rs. 20,000 for 1 yr = Rs. [3500 – 1200 – 600]

= Rs. 1700

∴The remaining amount should be invested at 8.5% per annum.

1.8 COMPOUND INTEREST

To define an interest rate fully, allowing comparisons with other interest rates, both the interest rate and the compounding frequency must be disclosed. Since most people prefer to think of rates as a yearly percentage, many governments require financial institutions to disclose the equivalent yearly compounded interest rate on deposits or advances. For instance, the yearly rate for a loan with 1% interest per month is approximately 12.68% per annum ($1.01^{12} - 1$). This equivalent yearly rate may be referred to as *annual percentage rate* (APR), *annual equivalent rate* (AER), *effective interest rate*, *effective annual rate*, and other terms. When a fee is charged up front to obtain a loan, APR usually counts that cost as well as the compound interest in converting to the equivalent rate. These government requirements assist consumers in comparing the actual costs of borrowing more easily. For any given interest rate and compounding frequency, an equivalent rate for any different compounding frequency exists. Compound interest may be contrasted with simple interest, where interest is not added to the principal (there is no compounding). Compound interest is standard in finance and economics, and simple interest is used infrequently (although certain financial products may contain elements of simple interest).

1.8.1 Terminology: The effect of compounding depends on the frequency with which interest is compounded and the periodic interest rate which is applied. Therefore, to accurately define the amount to be paid under a legal contract with interest, the frequency of compounding (yearly, half-yearly, quarterly, monthly, daily, etc.) *and* the interest rate must be specified. Different conventions may be used from country to country, but in finance and economics the following usages are common:

The *periodic rate* is the amount of interest that is charged (and subsequently compounded) for each period divided by the amount of the principal. The periodic rate is used primarily for calculations and is rarely used for comparison.

The *nominal annual rate or nominal interest rate* is defined as the periodic rate multiplied by the number of compounding periods per year. For example, a monthly rate of 1% is equivalent to an annual nominal interest rate of 12%.

The *effective annual rate* is the total accumulated interest that would be payable up to the end of one year divided by the principal.Economists generally prefer to use effective annual rates to simplify comparisons, but in finance and commerce the nominal annual rate may be quoted. When quoted together with the compounding frequency, a loan with a given nominal annual rate is fully specified (the amount of interest for a given loan scenario can be precisely determined), but the nominal rate cannot be directly compared with that of loans that have a different compounding frequency.Loans and financing may have charges other than interest, and the terms above do not attempt to capture these differences. Other terms such as annual percentage rate andannual percentage yield may have specific legal definitions and may or may not be comparable, depending on the jurisdiction.

When interest is calculated on the Principal for the entire period of loan, the interest is called simple interest and is given by:

I= P*R*T/100

But if this interest is due (not paid) after the decided time period, then it becomes a part of the principal and so is added to the principal for the next time period, and the interest is calculated for the next time period on this new principal. Interest calculated, this way is called compound interest. The time period after which the interest is added to the principal for the next time period is called the Conversion Period.

The conversion period may be one year, six months or three months and the interest is said

to compounded, annually, semi-annually or quarterly, respectively.

In other words, if you walk into a bank and open up a savings account you will earn interest on the money you deposit in the bank. If the interest is calculated once a year then the interest is called “simple interest”. If the interest is calculated more than once per year, then it is called “compound interest”. The mathematical formula for calculating compound interest depends on several factors. These factors include the amount of money deposited called the principal, the annual interest rate (in decimal form), the number of times the money is compounded per year, and the number of years the money is left in the bank.

Example1: Find the compound interest on a sum of Rs. 2000, for two years when the
interest is compounded annually at 10% per annum.

Sol: Here P = Rs. 2000 and R = 10%

∴ Interest for the first conversion time period (i.e. first year)

= 2000*10*1/100 = Rs. 200

∴ Principal for the second year (or 2nd conversion period)

= Rs. (2000 + 200) = Rs. 2200

∴ Interest for the 2nd time period = Rs. 2200*10*1/100 = Rs. 220

∴ Amount payable at the end of two years = Rs. (2200 + 220)

= Rs. 2420

∴ Total interest paid at the end of two years = Rs. (2420 – 2000)

= Rs. 420

or [Rs. (200 + 220) = Rs. 420]

∴ Compound interest = Rs. 420

Thus, for calculating the compound interest, the interest due after every conversion period
is added to the principal and then interest is calculated for the next period.

Formula for Compound Interest

Let a sum P be borrowed for n years at the rate of r% per annum, then

Interest for the first year = P*R*1/100 = PR/100

Amount after one year = Principal for 2nd year = P + PR/100

$$P\ (1+R/100)$$

Interest for 2nd year = $P\ (1+R/100) * R*1/100 = PR/100(1+R/100)$

Amount after 2 years = $P + PR/100 + PR/100(1+R/100)$

$$= P(1+R/100)^2 \text{ and so on.}$$

Amount after n years = $P\ (1+r/100)^n$

Thus, if A represents the amount and R represents r% or r/100

$A = P(1 + R)n$

and compound interest = $A – P = P\ (1 + R)n – P$

$= P[(1 + R)n –1]$ or $P(1+r/100) - P$

Simple interest and compound interest are equal for first year (first conversion period).

To solve compound interest problems, we need to take the given information at plug the information into the compound interest formula and solve for the missing variable. The method used to solve the problem will depend on what we are trying to find. If we are solving for the time, t, then we will need to use logarithms because the compound interest formula is an exponential equation and solving exponential equations with different bases requires the use of logarithms.

Example 2: Calculate the compound interest on Rs. 20,000 for 3 years at 5% per annum, when the interest is compounded annually.

Sol: Here P = Rs. 20,000, R = 5% and n =3

$\therefore$ CI = $P\ [(1 + R)n –1]$

= Rs. 20000[(1+ 5/100) – 1]

= Rs. 20000[(21/20) – 1]

= Rs. 3152.50

Example 3: Calculate the compound interest on Rs. 20,000 for 1-1/2 years at the rate of 10% per annum, when the interest is compounded semi-annually.

Sol: Here P = Rs. 20,000, R = 10% per annum

= 5% per half year

and n = 3 half years

∴ CI = P[(1 + R)n –1] = Rs. 20000[(1+5/100) – 1]

= 20000 [(9261/8000 – 1)]

= Rs. 3125.50

Example 4: Calculate the compound interest on Rs. 20,000 for 9 months at the rate of 4% per annum, when the interest is compounded quarterly.

Sol: Here P = Rs. 20,000, R = 4% per annum

= 1% per quarter of year

and n = 3/4 yrs = 3 quarters

∴ CI = P[(1 + R)$_n$ –1] = 20000[(1+1/100) – 1] = 20000 * 30301/ 100*100*100 = = Rs. 606.02

Example 5: calculate the amount and compound interest on Rs. 12000 for 1-1/2years at the rate of 10% per annum compounded annually.

Sol: Here P = Rs. 12000, R = 10% and n = 1-1/2 years

Since interest is compounded, annually, so, amount at the end of 1 year is given by:

A= $P(1+ r/100)^n$

A= $12000(1+ 10/100)^1$

= Rs. 13200

∴ Principal for next 6 months = Rs. 13200.

and Rate R = 10%/2 = 5%

A = $13200(1+ 5/100)^1$

=13200*21/20

= ` 13860

∴Amount after 1-1/2 years = Rs. 13860

And Compound interest = Rs. [13860 – 12000]

= Rs. 1860

Example 6: At what rate per cent per annum will Ron lends a sum of Rs. 2000 to Ben. Ben returned after 2 years Rs. 2205, compounded annually?
Sol : Let the required rate be R% per annum.

Here, A = Rs. 2205, P = Rs. 2000 and n = 2 years.

Using the formula $A = P(1 + R/100)^n$,

$2205 = 2000 \times (1 + R/100)^2$

$\Rightarrow (1 + R/100)^2 = 2205/2000 = 441/400 = (21/20)^2$

$\Rightarrow (1 + R/100) = 21/20$

$\Rightarrow R/100 = (21/20 – 1) = 1/20$

$\Rightarrow R = (100 \times 1/20) = 5$

Hence, the required rate of interest is 5% per annum.

Example 7: A man deposited Rs. 1000 in a bank. In return he got Rs. 1331. Bank gave interest 10% per annum. How long did he kept the money in the bank?
Sol: Let the required time be n years. Then,

amount = Rs. $\{1000 \times (1 + 10/100)^n\}$

= Rs. $\{1000 \times (11/10)^n\}$

Therefore, $1000 \times (11/10)^n = 1331$ [since, amount = Rs. 1331 (given)]

$\Rightarrow (11/10)^n = 1331/1000 = 11 \times 11 \times 11/ 10 \times 10 \times 10 = (11/10)^3$

$\Rightarrow (11/10)^n = (11/10)^3$

$\Rightarrow n = 3.$

Thus, $n = 3$.

Hence, the required time is 3 years.

1.9 BILL OF DISCOUNTING-BUSINESS APPLICATIONS

Bill discounting, as a fund-based activity, emerged as a profitable business in the early nineties for finance companies and represented a diversification in their activities in tune with the emerging financial scene in India. In the post-1992 (scam) period its importance has substantially declined primarily due to restrictions imposed by the Reserve Bank of India.

1.9.1 Concept

According to the Indian Negotiable Instruments Act, 1881: "The bill of exchange is an instrument in writing containing an unconditional order, signed by the maker, directing a certain person to pay a certain sum of money only to, or to the order of, a certain person, or to the bearer of that instrument." The bill of exchange (B/E) is used for financing a transaction in goods which means that it is essentially a trade-related instrument.

1.9.2 Definition and Meaning

When a buyer buys goods from the seller, the payment is usually made through letter of credit. The credit period may vary from 30 days to 120 days. Depending upon the credit worthiness of the buyer, the bank discounts the amount that needs to be paid at the end of credit period. It means that the bank will charge the interest amount for the credit period as an advance from the buyer's account. After that, the bill amount is paid as per the end of the time span with respect to the agreed upon document between the buyer and seller.

In other words, If the drawer of the bill does not want to wait till the due date of the bill and is in need of money, he may sell his bill to a bank at a certain rate of discount. The bill will be endorsed by the drawer with a signed and dated order to pay the bank. The bank will become the holder and the owner of the bill. After getting the bill, the bank will pay cash to the drawer equal to the face value less interest or discount at an agreed rate for the number of days it has to run. This process is known as discounting of a bill of exchange. In capitalist countries, the purchase of bills of exchange by banks before maturity. The bank pays less thannominal value of the bill, deducting a certain percentage for interest. At maturity, the bank collects the full nominal value from the drawee. The discounting of bills expands commercial credit and speeds up the circulation of capital. At the same time, the discounting of commercialpaper and especially of finance bills encourages speculation, increases the anarchy of capitalism, and has tens the onset of economic crises.In the age of the general crisis of capitalism, the proportion of bank assets represented by longterm loans and investments in secur ities ofvarious types has increased, while that represented by discounted bills has decreased. At the same time, the structure of discounting haschanged, with the discounting of treasury bills becoming increasingly important. The proceeds from the sale of these bills are used bygovernments for such purposes as financing military expenditures and covering budget deficits.

1.9.2 Types of Bills

There are various types of bills. They can be classified on the basis of when they are due for payment, whether the documents of title of goods accompany such bills or not, the type of activity they finance, etc.

Some of these bills are:

a. *Demand Bill* -This is payable immediately "at sight" or "on presentment" to the drawee. A bill on which no time of payment or "due date" is specified is also termed as a demand bill.
b. *Usance Bill* -This is also called time bill. The term usance refers to the time period recognized by custom or usage for payment of bills.
c. *Documentary Bills* - These are the B/Es that are accompanied by documents that confirm that a trade has taken place between the buyer and the seller of goods. These documents include the invoices and other documents of title such as railway receipts, lorry receipts and bills of

lading issued by custom officials. Documentary bills can be further classified as: (i) Documents against acceptance (D/A) bills and (ii) Documents against payment (DIP) bills. D/ A Bills In this case, the documentary evidence accompanying the bill of exchange is deliverable against acceptance by the drawee. This means the documentary bill becomes a clean bill after delivery of the documents. Clean Bills are not accompanied by any documents that show that a trade has taken place between the buyer and the seller. Because of this, the interest rate charged on such bills is higher than the rate charged on documentary bills. Creation of a B/E Suppose a seller sells goods or merchandise to a buyer. In most cases, the seller would like to be paid immediately but the buyer would like to pay only after some time, that is, the buyer would wish to purchase on credit. To solve this problem, the seller draws a B/E of a given maturity on the buyer. The seller has now assumed the role of a creditor; and is called the drawer of the bill. The buyer, who is the debtor, is called the drawee. The seller then sends the bill to the buyer who acknowledges his responsibility for the payment of the amount on the terms mentioned on the bill by writing his acceptance on the bill. The acceptor could be the buyer himself or any third party willing to take on the credit risk of the buyer. Discounting of a B/E The seller, who is the holder of an accepted B/E has two options: 1. Hold on to the B/E till maturity and then take the payment from the buyer. 2. Discount the B/E with a discounting agency. Option (2) is by far more attractive to the seller. The seller can take over the accepted B/E to a discounting agency bank, NBFC, company, high net worth individual] and obtain ready cash. The act of handing over an endorsed B/E for ready money is called discounting the B/E. The margin between the ready money paid and the face value of the bill is called the discount and is calculated at a rate percentage per annum on the maturity value. The maturity a B/E is defined as the date on which payment will fall due. Normal maturity periods are 30, 60, 90 or 120 days but bills maturing within 90 days seem to be the most popular.

1.9.3 Advantages:

The advantages of bill discounting to investors and banks and finance companies are as follows:

- **To Investors**

1. Short-term sources of finance;
2. Bills discounting being in the nature of a transaction is outside the purview of Section 370 of the Indian Companies Act 1956, that restricts the amount of loans that can be given by group companies;
3. Since it is not a lending, no tax at source is deducted while making the payment charges which is very convenient, not only from cash flow point of view, but also from the point
of view of companies that do not envisage tax liabilities;
4. Rates of discount are better than those available on ICDs;
5. Flexibility, not only in the quantum of investments but also in the duration of investments.

- **To Banks**

1. *Safety of Funds*- The greatest security for a banker is that a B/E is a negotiable instrument bearing signatures of two parties considered good for the amount of bill; so he can enforce his claim easily.
2. *Certainty of Payment* -A B/E is a self-liquidating asset with the banker knowing in advance the date of its maturity. Thus, bill finance obviates the need for maintaining large, unutilised, ideal cash balances as under the cash credit system. It also provides banks greater control over their drawls.
3. *Profitability*- Since the discount on a bill is front ended, the yield is much higher than in other loans and advances, where interest is paid quarterly or half yearly.
4. *Evens out Inter-Bank Liquidity Problems* -The development of healthy parallel bill discounting market would have stabilized the violent fluctuations in the call money market as banks could buy and sell bills to even out their liquidity mismatches.

Discount Rate and Effective Rate of Interest Banks and finance companies discounting bills prefer to discount LlC (letter of credit)-backed bills compared to clean bills. The rate of discount applicable to clean bills is usually higher than the rate applicable to LlC-based bills. The bills are generally discounted up-front, that is, the discount is payable in advance. As a consequence, the effective rate of interest is higher than the quoted rate (discount). The discount rate varies from time to time depending upon the short-term interest rate.

The computation of the effective rate of interest on bills discounting is shown in following illustration

E**xample:**

The Hypothetical Finance ltd. discounts the bills of its clients at the rate specified below:

- L/C - backed bills, 22 per cent per annum
- Clean bill, 24 per cent per annum

Required: Compute the effective rate of interest implicit in the two types of bills assuming usance period of (a) 90 days for the L/C - based bill and (b) 60 days for the clean bill and value of the bill, Rs 10,000.

Sol. Effective Rate of Interest on L/C - based Bill:

Value of the bill, Rs 10,000

Discount charge Rs. 550 (Rs 10,000x 0.22 x 90/360)

Amount received by the client, = Rs 9,450 (Rs 10,000 - Rs 550)

Quarterly effective interest rate = 5.82% [Rs 90 x 100/Rs. 9450]

Annualised effective rate of interest, [(1.0582)4- 1] x 100 = 25.39%

Effective Rate of Interest on Clean Bill:

Value of bill Rs 10,000

Discount Charge, = Rs 400 (Rs 10,000 x 0.24 x 60/360)

Amount received by the client = Rs 9,600 (Rs 10,000 - Rs 400)

Quarterly rate of interest = 4.17% (Rs. 400/Rs 9,600) x100

Effective rate of interest per annum, = 17.75%.

1.9.4 Present Position of Bills Discounting

Financial services companies had been acting till the early nineties as bill-brokers for sellers and buyers of bills arising out of business transactions. They were acting as link between banks and business firms. At times they used to take up bills on their own account, using own funds or taking short-term accommodation from banks working as acceptance/discount houses. They had been handling business approximating Rs 5,000 crores annually. Bill discounting, as a fund-based service, made available funds at rates I per cent lower than on ash credit finance and bill finance constituted about one-fourth of bank finance. However, the bill re-discounting facility was misused by banks as well as the bill-brokers. The Jankiraman Committee appointed by the RBI which examined the factors responsible for the securities-scam identified the following misuse of the scheme: Banks have been providing bill finance outside the consortium without informing the consortium bankers;

1.9.5 Bills of discounting is of two types

1. Purchase bills discounting and
2. Sales bill discounting.

A purchase bill discounting means that the investor discounts the purchase bill of the company and pays the company, who in turn pay their supplier. The investor gets his money back from the company at the end of the discounting period. A sales bills discounting means the investor discounts the sales bill of the company and pays directly to the company. The investor gets his return from the company at the end of the discounting period.

Funds -The funds generally required for this type of transaction start from Rs300, 000 to upto Rs2 mn. The tenure, generally, ranges from 60 days to upto 180 days.

Procedure- The procedure is that a broker will contact you with proposals to discount bills of different companies at different rates of discounting. The better companies command discounting rates of 13% to 15%, while the lesser known, by size and by safety, have to pay discounting rates of 17% to as high as 28%. It is later explained what factors determine the discount rates. When an investor and the company agree to a particular bill discounting transaction, the following is what the company gives to the investor: The original copies of bills to be discounted; A hundi / promissory note; Post dated cheque. The investor simply has to issue a cheque. The amount of cheque is arrived at after deducting the discount rate. The post dated cheque that the company gives is of the full amount of the transaction. This can be better explained with an example as follows.

Company A wants to discount its purchase bill of Rs200, 000 for a period of three months. Investor P agrees to do so at a discount rate of 21%. The deal is mutually agreed. Now, the investor will issue a cheque of Rs189, 500.

This figure is arrived at as follows:

= 200,000 x 21% x 3/12

Thus the investor gets his/her interest before the end of the period on discounting. The company on its part will issue a post-dated cheque of Rs200, 000 for three months period. Here we see that the investor benefits in two ways: He gets the interest element at the first day of issuing the cheque. i.e. he does not include that part in his cheque amount.

Thus he can earn interest on this interest for a three-month period. Even a simple bank fixed deposit on it will earn @5% p.a. By investing Rs189, 500 for three months, the investor earns Rs10, 500 on it. A return of 22. 16%.

Discount Rates- The rates depend on the following factors: The Broker: The broker has a good influence on the rates offered by companies. His relations with the company and the investor do make a difference of a couple of percentage point in discounting rates. Market Liquidity: Liquidity crunch in the market tends to hike up the rates even in the best of the companies. Since this instrument is a short tenure one, short-term changes in the market liquidity greatly affect the discount rates. Volume/Value of Discounting: When the volume/value of discounting done by the investor is high, he is looking at security more than returns. The company on its part is looking at savings by way of reduced legal paper work and a higher amount of dedicated funds for a said period and hence on the whole reduced costs to the company. Frequency: An investor who is regular bills discounter for the company may get upto 1% to 1.5% points higher interest rates than a new investor. As for the investor he is trying it out with a new company and will agree to a lesser rate to ensure safety. Company’s finance resources: This is one of the biggest factors that decide the discount rates. A

Public limited company generally tends to have a cheaper source of finance as against any other form of company. Working capital financing of companies to a large extent manipulates the rates the companies are willing to discount their bills at.

Caveat- The following points need to be remembered when dealing in this instrument. One must have a thorough knowledge of the company whose bills are discounted. Their industry, competition, people at the helm and their reputation in the market. This is necessary as it is going to be the company that is going to pay you from its earnings. There is no legal fall back option in case of default by the company. The company does sign a promissory note, but legal respite using this will take years to happen. The investor is not a secured creditor for the company nor does he get any preference on winding up of the company. Brokers need to be people who are well known to you. Since most of the deals happen through them, you should know the broker well enough to trust him and his deals. Spurious brokers are plenty out there in the market and a watchful eye must be kept. Even after investing in the company a regular watch on the company's fortune and a constant touch with the broker would be warranted.

1.10 SUMMARY

Business mathematics is mathematics used by commercial enterprises to record and manage business operations. Commercial organisations use mathematics in accounting, inventory management, marketing, sales forecasting, and financial analysis. Mathematics typically used in commerce include elementary arithmetic, elementary algebra, statistics and probability. Business management can be made more effective in some cases by use of more advanced mathematics such as calculus, matrix algebra and linear programming.

Business Mathematics is very important for modern business management. The forecasting and operating procedures are based primarily on business mathematics. Things such as simple interest and compound interest show a company what it will lose or get over the years if it invests in a particular asset. Business mathematics also helps in cost and price calculations which are the basis of cash inflows and outflows that all companies have to deal with.

Equated Monthly Instalment (EMI) refers to the monthly payment a borrower makes on his loan. Though it is a combination of interest payment and principal repayment, the total monthly amount is calculated in such a way that it remains constant all through the repayment tenure. The below mentioned example is given to explain the methodology. Consider that a loan of Rs.1,00, 000 is to be repaid over 25 years in equal monthly installments. If the annual interest rate is 7%, the monthly EMI is calculated as follows: (Since the repayments are monthly). So, The EMI will be fixed as Rs.710 in this case by rounding off to the nearest 10 rupees. To determine the specific percent of a number or quantity, we change the percent to a fraction or a decimal and then multiply. When the selling price is more than the cost price of the goods, there is a profit (or gain). When the selling price is less than the cost price of the goods, there is a loss. The simple interest (I.) on a principal (P) at the rate of R% for a time T years, is calculated, using the formula:

$$I = P \times R \times T$$

Discount is a reduction in the list price of goods. Discount is always calculated on the marked price of the goods. (Marked price – discount), gives the price, which a customer has to pay while buying an article. A discount series can be reduced to a single discount. Sales tax is charged on the sale price of goods. An instalment plan enables a person to buy costlier goods. In the case of compound interest. Amount (A) = P (1 + R)n, where P is the Principal, R = rate% and n = time. Compound interest is greater than simple interest, except for the first conversion period. The interest on loans and mortgages that are amortized-that is, have a smooth monthly payment until the loan has been paid off-is often compounded monthly.

1.11 SELF ASSESSMENT EXERCISES

1. Bobby Cash deposited Rs 10,000 at 8% compounded quarterly. Two years after she makes the first deposit, he adds another Rs. 20,000,

also at 8% rate compounded quarterly. What total amount will he have 4 years after his first deposit?

2. For one year, a student loan of Rs. 52,000 at 9% compounded semi-annually resulted in a maturity value of Rs. 5,934.06.

3. Billy Jean King deposited Rs. 6500 in an account paying 7.5 % compounded quarterly. After 3 years the rate drops to 4% compounded semi-annually. Find the amount in her account at the end of 7 years.

4. John and Jill have Rs. 20,000 cash for the down payment of a house and they can afford a 15-year mortgage payment of Rs. 2,500/month. If the best mortgage rate that they can get is 7.5% then what will be the most affordable home that they can buy by their current budget plan?

5. If money can be borrowed at 8 % compound monthly, which one is larger: Rs. 10,000 now or Rs. 15,000 in 5 years? Use present value to decide.

6. Mention the advantages of quantitative approach to management.

7. What factors in modern society contribute to the increasing importance of quantitative approach to management?

8. A retailer buys a cooler for Rs. 3800 but had to spend Rs. 200 on its transport and repair. If he sells the cooler for Rs. 4400, determine, his profit percent.

9. A vendor buys lemons at the rate of 5 for Rs. 7 and sells them at Rs. 1.75 per lemon. Find his gain percent.

10. A man purchased a certain number of oranges at the rate of 2 for Rs. 5 and sold them at the rate of 3 for Rs. 8. In the process, he gained Rs. 20. Find the number of oranges he bought.

11. By selling a bi-cycle for Rs. 2024, the shopkeeper loses 12%. If he wishes to make a gain of 12% what should be the selling price of the bi-cycle?

12. By selling 45 oranges for Rs. 160, a woman loses 20%. How many oranges should she sell for Rs. 112 to gain 20% on the whole?

13. A dealer sold two machines at Rs. 2400 each. On selling one machine, he gained 20% and on selling the other, he lost 20%. Find the dealer's net gain or loss percent.

14. Harish bought a table for Rs. 960 and sold it to Raman at a profit of 5%. Raman sold it to Mukul at a profit of 10%. Find the money paid by Mukul for the table.

15. A man buys bananas at 6 for Rs. 5 and an equal number at Rs. 15 per dozen. He mixes the two lots and sells them at Rs. 14 per dozen. Find his gain or loss percent, in the transaction.

16. If the selling price of 20 articles is equal to the cost price of 23 articles, find the loss or gain percent.

17. Rama borrowed Rs. 14000 from her friend at 8% per annum simple interest. She returned the money after 2 years. How much did she pay back altogether?

18. Ramesh deposited Rs. 15600 in a financial company, which pays simple interest at 8% per annum. Find the interest he will receive at the end of 3 years.

19. Naveen lent Rs. 25000 to his two friends. He gave Rs. 10,000 at 10% per annum to one of his friend and the remaining to other at 12% per annum. How much interest did he receive after 2 years.

20. Shalini deposited Rs. 29000 in a finance company for 3 years and received Rs. 38570 in all. What was the rate of simple interest per annum?

21. At what rate of interest will simple interest be half the principal in 5 Years.

22. A sum of money amounts to Rs. 1265 in 3 years and to Rs. 1430 in 6 years, at simple interest. Find the sum and the rate percent.

23. Out of Rs. 75000 to invest for one year, a man invested Rs. 30000 at 5% per annum and Rs. 24000 at 4% per annum. At what percent per annum, should he invest the remaining money to get 6% interest on the whole money?

24. A certain sum of money doubles itself in 8 years. In how much time will it become 4 times of itself at the same rate of interest?

25. Calculate the compound interest on Rs. 16000 for 9 months at 20% per annum, compounded quarterly.

26. Find the sum of money which will amount to Rs.27783 in 3 years at 5% per annum, the interest being compounded annually.

27. Find the difference between simple interest and compound interest for 3 years at 10% per annum, when the interest is compounded annually on Rs. 30,000.

28. A sum of money is invested at compound interest for 9 months at 20% per annum, when the interest is compounded half yearly. If the interest were compounded quarterly, it would have fetched Rs. 210 more than in the previous case. Find the sum.

29. A sum of Rs. 15625 amounts to Rs. 17576 at 8% per annum, compounded semi-annually. Find the time.

30. Find the rate at which Rs. 4000 will give Rs. 630.50 as compound interest in 9 months, interest being compounded quarterly.

31. A sum of money becomes Rs. 17640 in two years and Rs. 18522 in 3 years at the same rate of interest, compounded annually. Find the sum and the rate of interest per annum.

1.12 SUGGESTED READINGS

- Statistical Methods. Gupta S.P.
- Business Mathematics and Statistics. P.R. Vital
- Business mathematics. Kavita Choudhary.

STRUCTURE

2.1 INTRODUCTION

The set concept, fundamental in modern mathematics, was introduced by Georg Cantor in the nineteenth century. Cantor's concept profoundly influenced mathematical thinking, but most of his fellow mathematicians initially rejected his ideas. Everything about the concept was new and shocking. It contradicted the accepted way of thinking, especially with regard to infinity. However, after the mathematics community became familiar with the Cantorian concept, flaws were discovered in his set theory. Probably the best-known description of those flaws is found in the Russell paradox. Bertrand Russell made a successful attempt to solve the flaws within the Cantorian set theory by elaborating a new approach. (With regard to the Russell set theory and the Russell paradox, see Wilder, 1965.) Nowadays the concept of set is an integral part of mathematics, and is one of its central, seminal components.

In our mathematical language, everything in this universe, whether living or non-living, is called an object. If we consider a collection of objects given in such a way that it is possible to tell beyond doubt whether a given object is in the collection under consideration or not, then such a collection of objects is called a well defined collection of objects.

The term 'set' is introduced in mathematics without an initial definition. It is rather described as a collection. Nevertheless, a number of properties are established especially in order to distinguish the mathematical concept of set from the term 'collection' from which it is derived.

2.2 OBJECTIVES

This unit explains basic concepts in Set Theory, describing sets, elements, Venn diagrams and the union and intersection of sets. The specific objectives are:

- To know about the concept of sets.
- To understand the application of sets in business mathematics.

2.3 CONCEPT OF SETS

In school mathematics, the concept of set is used in various contexts but, generally, in an inconsistent manner. A major difficulty for students is the fact that the set concept is accepted in mathematics as a primitive, undefined concept (like, point, straight line, etc.) in contrast to the formal definition one starts with in an intuitive model, the idea of a collection of objects. But the mathematical set concept has a number of formal properties and aspects which contradict the initial collection model.

SET: A well defined collection of objects is called a set.

The objects in a set are called its members or elements.

We usually denote sets by capital letters and their elements by small letters.

If is an element of a set A, we write $x \in$ **A**, which means that, 'x belongs to **A'** or that 'x is an element of **A'.**

If x does not belong to **A**, we write, $x \notin$ **A**.

Ex.1. The collection of vowels in English alphabet is a set containing five elements, namely a, e, i, o, u.

Ex.2. The collection of first four prime numbers is a set containing the elements 2, 3, 5, 7.

Ex.3. The collection of all beautiful girls of India, is not a set, since the term ‘beautiful’ is vague and it is not well defined.

Similarly, ‘rich persons’; ‘honest persons’; ‘good players’ etc. do not form sets.

Ex.4. We denote the sets of all natural numbers, all integers; all rational numbers and all real numbers by N, I, Q and R respectively.

2.3.1 How to Describe or Specify a Set

There are two methods for describing a set.

(i) *Tabulation method or Roster form* (ii) *Rule method or Set builder form.*

I. ROSTER FORM: Under this method; we make a list of the elements of the set and put it within braces {}.

Ex.1. *If A is the set of vowels in English alphabet,*
then A = [a, e, i, o, u}.

Ex.2. *If B is the set of prime numbers less than 15,*
then B= {2, 3, 5, 7, 11,13}.

Ex.3. *If C is the set of all even numbers lying between* 10 *and* 500,
*then C = {*12, 14, 16, 18, ..., 494, 496, 498 }.

II. SET-BUILDER FORM: Under this method, we list the property or properties satisfied by the elements of the set. We write, (x : x satisfies properties P}.

This means,

'the set of all those x such that each x satisfies the properties P'.

Ex.1. If **A** = {1, 2, 3, 4, 5), then we can write,

A = { x e N : x < 6).

Ex.2. If **B** = { 1, 2, 3, 4, 6, 8, 12, 24 }, then we can write,

B = [x : x is a factor of 24 } .

2.3.2 Types of Set

1. **EMPTY SET**: A set consisting of no element at all is called an empty set or a null set or a void set and it is denoted by (J).

   In Roster form, ϕ is denoted by { }.

   Ex.1. $\{x : x \in N, 2 < x < 3\} = \phi$.

   Ex.2. $\{x : x \in R, x^2 = -1\} = \phi$.

   A set which has at least one element is called a non-empty set.

2. **SINGLETON SET:** A set consisting of a single element is called a singleton set. *e.g*. [0 } is a singleton set, whose only member is 0

3. **FINITE AND INFINITE SET:** A set in which the process of counting of elements surely comes to an end, is called a finite set, otherwise it is called an infinite set.

   Ex.1. Each one of the sets given below is a finite set*:*

   (i) set of all persons on the earth ;

   (ii) $\{x : x \in N, x < 5$ crores};

   (iii) {x: $x \in I$ *x* is a factor of 1000) .

   Ex.2. Each one of the sets given below is an infinite set:

   (i) set of all points on an arc of a circle;

   (ii) set of all concentric circles with a given centre ;

   (iii) $[x \in I : x < 1\}$;

   (iv) $\{x \in Q : 0 < x < 1\}$.

Cardinal number of a finite set: *The number of distinct elements contained in a finite set A is called its cardinal number, to be denoted by n* (A).

Ex. If A = (2, 3, 5, 7, 11}, then *n* (A) = 5.

4. **EQUAL SETS:** Two sets A and B are said to be equal, written as $A = B$, if every element of A is in B and every element of B is in A.

   Remarks, (i) The elements of a set may be listed in any order.

   Thus, (1, 2, 3} = { 2, 1, 3 } = { 3, 2, 1 } etc.

   (ii) The repetition of elements in a set is meaningless.

   Ex.1. { x : x is a letter in the word, 'follow'} = {f, o, I, w}.

   Ex.2. Show that ϕ, (0) and 0 are all different.

   Sol. Since ϕ is a set containing no element at all; {0} is a set containing one element, namely 0; And, 0 is a number, not a set.

   Hence, $\phi \neq \{0\} \neq 0$.

5. **EQUIVALENT SETS:** Two finite sets A and B are said to be equivalent, if n (A) = n (B).

   Clearly, equal sets are equivalent but equivalent sets need not be equal.

   Ex. A = {1, 3, 5 } and ***B*** = { 2, 4, 6 } are equivalent but not equal

6. **SUB SETS:** If A and B are two sets such that every element of A is in B, we say that A is a subset of B and write, $A \subseteq B$.

   C

   If $A \subseteq B$ and $A * B$, then *A* is called a proper subset of *B,* written as $A \subseteq B$.

   Remarks, (i) If there exists even a single element in *A,* which is not in *B*, then A is not a subset of *B* and we write, $A \not\subset B$.

   (ii) ϕ has no proper subset.

   Ex.1. {3} C {2, 3, 5). Type equation here.

   Ex.2. {1,2) $\not\subset$(2,3,5}.

Ex.3. $N \subset W \subset I \subset Q \subset B$. But, $W \subset N$ and $\{0\} \not\subseteq N$.

7. UNIVERSAL SET: If there are some sets under consideration, then there happens to be a set which is a superset of each one of the given sets. Such a set is known as the universal set, to be denoted by U.

Ex.1. If $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 5, 7\}$ and $C = \{2, 4, 6, 8\}$, then $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is the universal set.

Ex.2. When we discuss sets of lines, triangles or circles in two dimensional geometry, the plane in which these lines, triangles or circles lie, is the universal set.

8. POWER SET: The collection of all possible subsets of a given set A is called the power set of A, to be denoted by P (A).

Ex.1. If $A = \{1, 2, 3\}$, then

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Ex.2. If $A = \{1, \{2\}\}$, we may write, $A = \{1, B\}$, where $B = \{2\}$.

$\therefore P(A) = \{\phi, \{1\}, \{B\}, \{1, B\}\}$

$= \{\phi, \{1\}, \{\{2\}\}, \{1, \{2\}\}\}$.

Ex.3. If $A = \{1, \{2, 3\}\}$, we may write, $A = \{1, B\}$, where $B = \{2, 3\}$.

$\therefore P(A) = \{\phi, \{1\}, \{B\}, \{1, B\}\} = \{\phi, \{1\}, \{\{2, 3\}\}, \{1, \{2, 3\}\}\}$.

Ex.4. Write down the power sets of:

(i) ϕ (ii) $\{\phi, \{\phi\}\}$.

Sol. (i) $P(\phi) = \{\phi\}$.

(iii) Let $A = \{\phi, \{\phi\}\} = \{\phi, B\}$, where $B = \{\phi\}$.

$P(A) = \{\phi, \{\phi\}, \{B\}, \{\phi, B\}\}$ or

or $P(A) = \{\phi. \{\phi\}, \{B\}, \{\phi, (\phi)\}\}$.

Ex.5. Show that $n\{P[P(P(\phi))]\} = 4$.

Sol. $P(\phi) = \{\phi\} \Rightarrow P(P(\phi)) = \{\phi, \{\phi\}\}$

$= P[\{P(\phi))] = \{\phi, (\phi\}\}\}$

$n\{P[P(P(\phi))]\} = 4$.

2.3.3 SOME RESULTS ON SUBSETS

Theorem 1. Every set is a subset of itself.

Proof. Let A be any set. Then, each element of A is clearly, there in A Hence, $A \subseteq A$.

Theorem 2. The empty set is a subset of every set.

Proof. Let A be any set. In order to show that $\phi \subseteq A$, we must show that there is no element of $ which is not contained in A. And, since ϕ contains no element at all, no such element can be found out. Hence, $\phi \subseteq A$.

Theorem 3. The total number of all possible subsets of a given set containing n elements is 2^n.

Proof. Let A be any set containing n elements. Then, one of its subsets is the empty set. Apart from this,

the number of singleton subsets of A = $n = {}^nC_1$;

the number of subsets of A, each containing 2 elements = ${}^nC_2$;

the number of subsets of A, each containing 3 elements = ${}^nC_3$;

the number of subsets of A, each containing n elements = ${}^nC_n$.

Total number of all possible subsets of A

$= \{ 1 + {}^nC_1 + {}^nC_2 + {}^nC_3 + ...+ {}^nC_n)$

$= (1 + 1)^n = 2^n$. [Using binomial theorem]

2.4 OPERATION OF SETS

In ordinary arithmetic and algebra there are four common operations that can be performed; namely, addition, subtraction, multiplication and division. With sets, however, just two operations are defined. These are set union and set intersection. Both of these operations are described, with examples, in the following sections. The Union of two sets A and B is written as AUB and defined as that set which contains all the elements lying within either A or B or both. For example, if A = (c,d,f,h,j) and B = (d,m,c,f,n,p), then the union of A and B is AUB = (c,d,f,h,j,m,n,p), these being the elements that lie in either A

or B. So that any element of *A* must be an element of *A*U*B*; similarly any element of *B* must also be an element of *A*U*B*. Set union for three or more sets is defined in an obvious way.

2.4.1 UNION OF SETS: The union of two sets A and B, denoted by AUB is the set of all those elements, each one of which is either in A or in B or in both A and B,

Thus, AUB $= \{ x : x \in A$ or $x \in B \}$.

Clearly, $x \in A \cup B \Rightarrow x \in A$ or $x \in B$.

"And, $x \notin A \cup B \Rightarrow x \notin A$ and $x \notin B$.

Ex.1. *If A* = {1, 2, 3, 4} *and B* = {2, 4, 6, 8 }, *then*

AUB= {1,2,3,4,6,8}.

Ex.2. *If A* = {x: x *is a positive integer*}

and B = {x: x *is a negative intege*r} then, A U *B* = [*x* : x *is an integer, x*≠ 0}.

2.4.2 INTERSECTION OF SETS: The intersection of two sets A and B, denoted by $A \cap B$ is the set of all elements, common to both A and B.

Thus, $A \cap B = \{x: x \in A$ and $x \in B\}$.

Clearly, $x \in A \cap B = x \in A$ and $x \in B$.

And, $x \notin A \cap B = x \notin A$ or $x \notin B$.

Ex.1. Let A = {1, 2, 3, 4}, and B = { 2, 4, 6 }. Then, $A \cap B = \{2, 4\}$.

Ex.2. Let $A = \{x : x = 3n, n \in N\}$ and $B = \{x : x = 4n, n \in N)$.

Then, $A \cap B = \{x : x = 12n, n \in N \}$.

Disjoint Set: Two sets A and B are said to be disjoint, *if* $A \cap B = \emptyset$. If $A \cap B = \phi$ then A and B are said to be intersecting sets *or* overlaping sets.

Ex. if A = {1, 3, 5, 7, 9}, B = { 2,4, 6, 8} and C = { 2, 3, 5, 7, 11 }, then A and B are disjoint sets, while A and C are intersecting.

2.4.3 DIFFERENCE OF SETS: If A and B are two sets, then their difference A-B is defined by;

$A - B = [\, x : x \in A$ and $x \in B\,)$.

Thus, $x \in A - B \Rightarrow x \in A$ and $x \in B$.

Also, the symmetric difference $A \Delta B$ is defined by.

$A \Delta B = (A-B) \cup (B-A)$.

Ex. *Let A* = {1, 3, 5, 7, 9} *and B* = {2, 3, 5, 7, 11}.

Then, $A - B = \{1, 9\}$; $B - A = \{2, 11\}$.

And, $A \Delta B = (A-B) \cup (B-A) = \{l, 2, 9, ll\}$.

2.4.4 COMPLEMENT OF A SET: Let U be the universal set and let $A \subset U$.

Then, the complement of A, denoted by A' or U - A is defined as

$A' = (x \in U : x \notin A\}$.

Clearly, i.e $A' \Leftrightarrow x \notin A$.

Ex.1. *If U* = {1, 2, 3, 4, 5 } *and* A = { 2, 4 }, *then A'* = {1, 3, 5 }.

Ex.2. *Let U* = (x : x is a letter in English, alphabet) and

A = x : x is a vowel} , then A' = { x : x is a consonant}.

SOME RESULTS ON COMPLEMENTATION:

(i) $U' = \{x \in U : x \notin U\} = \phi$

(ii) $\phi = \{x \in U : x \notin \phi\} = U$

(iii) $(A')' = [\, x \in U : x \notin A'\} = [\, x \in U : x \in A\} = A$;

(iv) $A \cup A' = \{x \in U : x \cup A\} \cup \{x \in U : x \notin A) = U$;

(v) $A \cap A' = \{x \in U : x \in A\} \cap \{x \in U : x \notin A\} = \phi$

2.5 ALGEBRA OF SETS

Intuitively, a set is a “collection” of objects known as “elements.” But in the early 1900’s, a radical transformation occurred in mathematicians’ understanding of sets when the British philosopher Bertrand Russell identified a fundamental paradox inherent in this intuitive notion of a set. Consequently, in a formal set theory course, a set is defined as a mathematical object satisfying certain axioms. These axioms detail properties of sets and are used to develop an elegant and sophisticated theory of sets.

Definition: • denotes the **empty set** { }, which does not contain any elements.

• N denotes the set of **natural numbers** {1, 2, 3, . . . }.

• Z denotes the set of **integers** { . . ,−3,−2,−1, 0, 1, 2, 3, . . . }.

• Q denotes the set of **rational numbers** {p/q : p, q 2 Z with q 6= 0 }.

• R denotes the set of **real numbers** consisting of directed distances from adesignated point zero on the continuum of the real line

• C denotes the set of **complex numbers** { a + bi : a, b 2 R with i = p−1 }.

In this definition, various names are used for the same collection of numbers. For example, the natural numbers are referred to by the mathematical symbol "N," the English words "the natural numbers," and the set-theoretic notation "{1, 2, 3, . . .}." Mathematicians move freely among these different ways of referring to the same number system as the situation warrants. In addition, the mathematical symbols for these sets are "decorated" with the superscripts "¤," "+," and "−" to designate the corresponding sub-collections of nonzero, positive, and negative numbers, respectively. For example, applying this symbolism to the integers
Z = {. . . ,−3,−2,−1, 0, 1, 2, 3, . . .}, we have Z = {. . . ,−3,−2,−1, 1, 2, 3, . . .},
Z+ = {1, 2, 3, . . .}, Z− = {−1,−2,−3, . . .}.

I. Idempotent Laws:

(i) $A \cup A = A$ (ii) $A \cap A = A$.

Proof. (i) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$.

(ii) A ∩ A = x : x e A and x ∈ A} = [x : x ∈ A } =A.

II. Identity Laws:

(i) $A \cup \phi = A$ (ii)$A \cap U = A$.

Proof (i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\}$

$== \{x: x \in A\} = A$ [∵ ϕ has no element]

(ii) $A \cap U = \{x: x \in A \text{ and } x \in U\} = \{x: x \in A\} = A$.

III. Commutative; Laws:

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$.

Proof (i) $A \cup B = \{x : x \in A \text{ or } x \in B\}$

(ii) $A \cap B = \{x: x \in A \text{ and } x \in B\}$.

$= \{x : x \in B \text{ and } x \in A\} = B \cap A$.

IV. Associative Laws:

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$

Proof. Let x be an arbitrary element of $(A \cup B) \cup C$

Then, $x \in (A \cup B) \cup C \Rightarrow x \in (A \cup B)$ or $x \in C$.

$\Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C$

$\Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C)$

$\Rightarrow x \in A \text{ or } x \in (B \cup C)$

$\Rightarrow x \in A \cup (B \cup C)$.

$(A \cup B) \cup C \subseteq A \cup (B \cup C)$.

Similarly, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

Hence, $(A \cup B) \cup C = A \cup (B \cup C)$.

V. Distributive Laws:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We prove (ii) and leave (i) as an exercise,

(ii) $x \in A \cap (B \cup C) \Rightarrow x \in A$ and $x \in (B \cup C)$

$\Rightarrow x \in A$ and $(x \in B$ or $x \in C) \Rightarrow (x \in A \;\&\; x \in B)$

or $(x \in A \;\&\; x \in C)$

[$\because$ 'and' distributes 'or']

$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$.

$\therefore A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Similarly, $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Hence, $A \cap (B \cup C) = (A \cap B)$ O $(A \cap C)$.

VI.De-Morgan's Laws:

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$.

Proof. (i) $x \in (A \cup B)'$ = x <s A u B

$\Rightarrow x \in A$ and $x \in B \Rightarrow x \in A'$ and $x \in B' \Rightarrow x \in A' \cap B'$.

$\therefore (A \cup B)' \subset (A' n B')$.

Similarly, $(A' \cap B) \subset (A \cup B)'$.

Hence, (A u B)'= $(A' \cap B')$.

ii) Hint, $x \in A \cap B \Rightarrow x \in A$ or $x \in B$.

SOME MORE RESULTS ON OPERATIONS ON SETS:

Theorem 1: For any sets A and B, prove that:

(i) $A \cap B \subset A$ and $A \cap B \subset B$;

(ii) $A - B = A \cap B'$*;*

(iii) $(A - B) \cup B = A \cup B$;

(iv) $(A - B) \cap B = \phi$

(v) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

Proof: (i) $x \in A \cap B$ $\mathbf{x} \in A$ and $x \in B \Rightarrow x \in A$ *(surely)*

$\therefore A \cap B \subset A$.

Similarly, $A \subseteq B \subseteq B$.

(ii) $x \in (A - B) \Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \in A$ and $x \in B'$

$\Rightarrow x \in (A \cap B')$.

$\therefore (A - B) \subseteq (A \cap B')$.

Similarly, $(A \cap B') \subseteq (A - B)$.

Hence, $A - B = A \cap B'$.

(iii) $x \in (A - B) \cup B \Rightarrow x \in (A - B)$ or $x \in B$ _

$\Rightarrow (x \in A \;\&\; x \in B)$ or $x \in B$

$\Rightarrow (x \in A$ or $x \in B)$ & $(x \notin B$ or $x \in B)$

$\Rightarrow x \in (A \cup B)$.

$\therefore (A - B) \cup B \subseteq (A \cup B)$ ...(I)

Again, $y \in (A \cup B)$

$\Rightarrow y \in A$ or $y \in B$

$\Rightarrow (y \in A$ or $y \in B)$ & $(y \notin B$ or $y \in B)$

[Note]

$(y \in A \;\&\; y \notin B)$ or $y \in B$

$\Rightarrow y \in (A - B)$ or $y \in B$

$\Rightarrow y \in (A - B) \cup B$.

$\therefore A \cup B \subseteq (A - B) \cup B$ ...(II)

Hence, from (I) & (II), we have: $(A - B) \cup B = (A \cup B)$. ,

(iv) If possible, let $(A - B) \cap B$ ϕ and let $x \in (A-B) \cap B$.

Then, $x \in (A - B) \cap B \Rightarrow x \in (A - B)$ and $x \in B$

$\Rightarrow (x \in A$ and $x \notin B)$ and $\boldsymbol{x} \in B$

$\Rightarrow x \in A$ and $(x \notin B$ and $x \in B)$.

But, $x \notin B$ and $\boldsymbol{x} \in B$ both can never hold simultaneously. Thus, we arrive at a contradiction.

Since the contradiction arises by assuming that $(A - B) \cap B \neq .\phi$, so $(A - B) \cap B = \phi$

(v) $x \in (A - B) \cup (B - A)$

$\Rightarrow (x \in A \;\&\; x \in B)$ or $(x \in B \;\&\; x \in A)$

=> ($x \in A$ or $x \in B$) and ($x \notin B$ or $x \notin A$)

=> $x \in (A \cup B)$ and $x \in (A \cap B)$

=> $x \in (A \cup B) - (A \cap B)$.

$\therefore (A-B) \cup (B-A) \subseteq (A \cup B) - (A \cap B)$.

Similarly, $(A \cup B) - (A \cap B) \subseteq (A-B) \cup (B-A)$.

Hence, $(A-B) \cup (B-A) = (A \cup B) - (A \cap B)$

Theorem 2. For any sets A and B, prove that:

(i) $A \subseteq B <=> B' \subseteq A'$;

(ii) $A - B = A \;\&\; A \cap B = \phi$

(iii) $A \cup B = A \cap B <=> A = B$

Proof, (i)

To prove: $A \subseteq B$ ϕ $B' \subseteq A'$.

We shall prove it in two parts, namely:

$(A \subseteq B => B' \subseteq A')$ and $(B' \subseteq A' => A \subseteq B)$

First part	***Second part***
Given : $A \subseteq B$. *To prove* : $B' \subseteq A'$. *Proof.* $x \in B' \Rightarrow x \notin B$ $\Rightarrow x \notin A$ [$\because A \subseteq B$] $\Rightarrow x \in A'$. $\therefore B' \subseteq A'$. **Thus,** $A \subseteq B \Rightarrow B' \subseteq A'$ **...(I)**	*Given* : $B' \subseteq A'$. *To prove* : $A \subseteq B$. *Proof.* $x \in A \Rightarrow x \notin A'$ $\Rightarrow x \notin B'$ [$\because B' \subseteq A'$] $\Rightarrow x \in B$. $\therefore A \subseteq B$. **Thus,** $B' \subseteq A' \Rightarrow A \subseteq B$ **...(II)**
Hence, from (I) & (II), we have $A \subseteq B \Leftrightarrow B' \subseteq A'$.	

(ii) **To prove:** $A-B = A \Leftrightarrow A \cap B = \phi$

We prove it into two parts, namely:

$[A - B = A => A \cap B = \phi]$ and $[A \cap B = \phi \Rightarrow A - B = A]$.

First part	*Second part*
Given **: $A - B = A$.**	***Given*** **: $A \cap B = \phi$.**
To prove **: $A \cap B = \phi$.**	***To prove*** **: $A - B = A$.**
Proof. **If possible, let $A \cap B \neq \phi$ and let $x \in A \cap B$. Then,**	***Proof.*** **Let $x \in A - B$. Then,**
$x \in A \cap B \Rightarrow x \in A$ and $x \in B$	**$x \in A - B \Rightarrow x \in A$ & $x \notin B$**
$\Rightarrow x \in (A - B)$ & $x \in B$	**$\Rightarrow x \in A$ (surely)**
$[\because A = A - B]$	**$\therefore (A - B) \subseteq A$.**
$\Rightarrow (x \in A$ & $x \notin B)$ & $x \in B$	**Again, let $y \in A$. Then,**
$\Rightarrow x \in A$ & $(x \notin B$ & $x \in B)$	**$y \in A \Rightarrow y \in A$ & $y \notin B$**
But, $x \notin B$ and $x \in B$ both can never hold simultaneously.	**$[\because A \cap B = \phi]$**
Thus, we arrive at a contradiction.	**$\Rightarrow y \in (A - B)$.**
$\therefore A \cap B = \phi$.	**$\therefore A \subseteq (A - B)$.**
Hence, $A - B = A \Rightarrow A \cap B = \phi$.	**Thus, $(A - B) = A$.**
	$\therefore A \cap B = \phi \Rightarrow (A - B) = A$.

Hence, $A - B = A \Leftrightarrow A \cap B = \phi$.

To prove: (iii) $(A \cup B) = (A \cap B) \Leftrightarrow A = B$.

We prove it in two parts, namely:

$[A \cup B = A \cap B \Rightarrow A = B]$ and $[A = B \Rightarrow A \cup B = A \cap B]$.

First part	*Second part*
Given: $A \cup B = A \cap B$.	Given : $A = B$
To prove: A-B.	To prove: $A \cup B = A \cap B$.
Proof. Let $x \in A$. Then,	**Proof**: $A \cup B = B \cup B = B$
$x \in A \Rightarrow x \in A \cup B$ $[\because A \subseteq A \cup B]$	$[\because A = B]$
$\Rightarrow x \in A \cap B$	And, $A \cap B = B \cap B = B$.
$[\therefore A \cup B = A \cap B]$	$[\because A = B]$
$\Rightarrow x \in A$ and $x \in B$	$\therefore A \cup B = A \cap B$ [each equal to B]
$\Rightarrow x \in B$ *(surely)*	
$\therefore A \subset B$.	Thus, $A = B \Rightarrow A \cup B = A \cap B$.
Similarly, $B \subset A$.	

∴ $A = B$. Thus, $A \cup B = A \cap B \Rightarrow A = B$.	
Hence, $A \cup B = A \cap B \Longrightarrow A = B$.	

Theorem 3. For any sets A, B, C prove that:

(i) $A - (B \cap Q = (A - B) \cup (A - C)$;

(ii) $A - (B \cup O = (A - B) \cap (A - C)$;

(iii) $A \cap (B - C) = (A \cap B) - (A \cap C)$;

(iv) $A \cap (B \,\Delta\, C) = (A \cap B) \,\Delta\, (A \cap C)$.

Proof, (i) Let $x \notin A - (B \cap C)$.

Then, $x \in A - (B \cap C)$

$\Rightarrow x \in A$ and $x \notin (B \text{ n } C)$

$\Rightarrow x \in A$ and $(x \notin B$ or $x \notin C)$

$\Longrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in (A - B)$ or $x \in (A-C)$

$\Rightarrow x \in (A-B) \cup (A-C)$.

∴ $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.

Similarly, $(A-B) \cup (A - C) \subseteq A - (B \cap C)$.

Hence, $A - (B \cap C) = (A - B) \cup (A - C)$.

Similarly, (ii) may be proved,

(iii) Let $x \in A \cap (B - C)$ .Then,

$x \in A \cap (B-C) \Rightarrow x \in A$ and $x \in (B - C)$

$\Rightarrow x \in A$ and $(x \in B$ and i.e $C)$

$\Rightarrow (x \in A \;\&\; x \in B) \;\&\; (x \in A \;\&\; x \Rightarrow C)$

$\Rightarrow x \Rightarrow (A \cap B) \;\&\; x \notin (A \cap C)$

$\Rightarrow x \in (A \cap B) - (A \cap C)$.

$\Rightarrow A \cap (B-C) \text{ c } (A \cap B) - (A \text{ n } C)$.

Similarly, $(A \cap B) - (A \cap C) \subseteq A \cap (B-C)$.

Hence, $A \cap (B-C) = (A \cap B) - (A \cap C)$.

(iv) $A \cap (B \Delta C)$

$= A \cap [(B-C) \cup (C-B)]$

$= [A \cap B-C)] \cup [A \cap (C-B)]$ [distributive law]

$= [(A \cap B)-(A \cap C)] \cup [(A \cap C)-(A \cap B)]$

$= (A \cap B) \Delta (A \cap C)$.

2.5.1 EULER-VENN DIAGRAMS

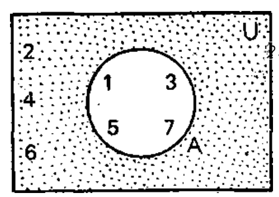
To express the relationship among sets in a perspective way, we represent them pictorially by means of diagrams, known as Venn-diagrams. The universal set is usually represented by a rectangular region and its subsets by closed bounded regions inside this rectangular region.

2.5.1.1 Venn diagrams in different situations:

I. Given: $A \subseteq U$, where U is the universal set.

Let $U = (1, 2, 3, 4, 5, 6, 7\}$.

And, $A = (1, 3, 5, 7\}$



Then, the rectangular region shown in Fig. 1 represents the universal set and the region enclosed by a closed curve inside the rectangular region represents the set A.

Clearly, the shaded region represents A'. So, A' = (2, 4, 6 }.

II. Given: Two intersecting subsets of a universal set.

Let U = { 1, 2, 3, 4, 5, 6, 7, 8 } be the universal set and

Let A = {1, 3, 4, 5} and B = { 2, 4, 5, 6 } be its two subsets.

As shown in Fig. 2, the rectangular region represents the universal set.

Since the sets A and B are intersecting, we draw two intersecting circles (or bounded figures), representing A and B respectively. Then

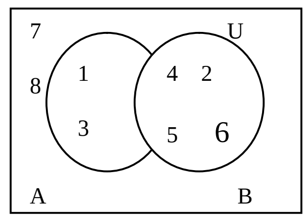
(i) The common region between the two circles represents $A \cap B$. Thus, $A \cap B = \{4, 5\}$.

(ii) The total region bounded by both A and B represents $A \cup B$. Thus, $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

(iii) Excluding the portion of B from the region of A, we get the region representing (A - B). Thus, $A - B = \{1, 3\}$.

(iv) Excluding the portion of A from the region of B, we get the region representing (B - A).

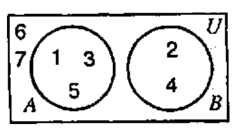
Thus, $B - A = \{2, 6\}$.



III. Given : Two disjoint subsets of a universal set. Let *U* = 1, 2, 3, 4, 5, 6, 7 } be the universal set and let A = { 1, 3, 5 } & B = {2, 4} be two of its subsets. Clearly, A and *B* are disjoint:

Since the sets A and *B* have no common element, we represent them by two disjoint circles, as shown in Fig. 3.

Clearly, $A \cap B = \phi$, $A - B = A$, $B - A = B$ & $A \cup B = \{1, 2, 3, 4, 5)$



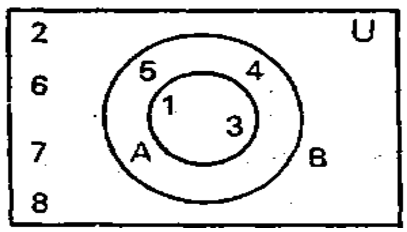
IV. Given: $A \subseteq B \subseteq U$, where U is the universal set. Let U =

{1, 2, 3, 4, 5, 6, 7, 8} be the universal set and let A = {1, 3 } and

B = {1, 3, 4, 5 } be two of its subsets. Since $A \subseteq B$, the region representing A must therefore lie wholly inside the region representing B, as shown in Fig. 4.

Clearly, $A \cap B = A$; $A \cup B = B$;

$A - B = \phi$ and $B - A = \{4, 5\}$.

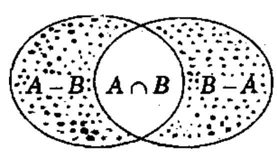


2.5.1.2 Important Results from Venn Diagrams

Let A and B be two nonempty intersecting sets. In counting the elements of $A \cup B$, the elements of $A \cap B$ are counted twice, once in the counting of elements of A and second time in the counting of elements of B.

Thus, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. If $A \cap B = \phi$,

Then $n(A \cup B) = n(A) + n(B)$.



It is also clear from the given Venn-diagram that:

(i) $n(A-B) + n(A \cap B) = n(A)$;

(ii) $n(B-A) + n(A \cap B) = n(B)$;

(iii) $n(A \cup B) = n(A-B) + n(A \cap B) + n(B-A)$.

Ex.1. A survey shows that 74% of the Indians like apples, whereas 68% *like* oranges. What percentage of the Indians like both apples and oranges?

Sol. Let A and B denote the sets of Indians who like apples and oranges respectively.

Then, $n(A) = 74$; $n(B) = 68$ *and* $n(A \cup B) = 100$.

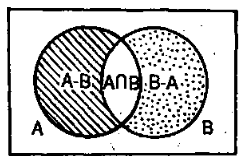
$\therefore$ $n(A \cap B) = n(A) + n(B) - n(A \cup B)$. $= (74 + 68 - 100) = 42$. Hence, 42% of the Indians like both apples and orange.

Ex. 2. In a group of 850 persons, 600 can speak Hindi and 340 can speak Tamil.

Find :(i) How many can speak both Hindi and Tamil;

(ii) How many can speak Hindi only ;

(iii) How many can speak Tamil only.



Sol. Let A and B denote the sets of persons who can speak Hindi and Tamil respectively. Then, $n(A) = 600$; $n(B) = 340$ & $n(A \cup B) = 850$:

(i) $n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= (600 + 340 - 850) = 90$. Thus, 90 persons can speak both Hindi and Tamil.

(ii) $n(A - B) + n(A \cap B) = n(A)$.

$\therefore n(A-B) = n(A) - n(A \cap B)$

$= (600 - 90) = 510$.

Thus, 510 persons can speak Hindi only.

(iii) $n(B - A) + n(A \cap B) = n(B)$.

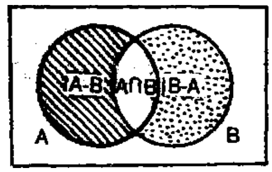
(iv) $n(B - A) = n(B) - n(A \cap B)$

$= (340 - 90) = 250$.

Thus, 250 persons can speak Tamil only.

Ex.3. In a group of 52 persons, 16 drink tea but not coffee and *33* drink tea. Find: (i) how many drink tea and coffee both; *(ii)* how many drink coffee but not tea.

Sol. Let A and B be sets of persons who drink tea and coffee respectively. Then (A - B) is the set of persons who drink tea but not coffee. And, (B -A) is the set of persons who drink coffee but not tea.



$\therefore$ n(A) = 33,n (A-B) = 16 and n(AUB) = 52.

Now, (i) n (A-B) + n (A ∩ B) = n (A)

$\therefore$ n(A∩B) = n(A)-n(A-B) = (33 - 16) = 17. Thus, 17 persons drink tea and coffee both,

(iii)n (AUB) = n (A) + n (B)-n (A∩B).

$\therefore$ n(B) = n(AuB) -n (A) + n(A∩B)

So, n (B) = (52 - 33 + 17) = 36. Now, n(B-A) + n(A∩B) = *n(B)*.

n(B-A) = n(B)-n(A∩B) = (36 - 17) = 19.

Hence, 250 persons drink coffee but not tea.

Ex.4. For any three sets A, B, C prove that :

n(AUBUC) = n(A) + n (B)+ n (C) + n (A ∩ B ∩ C)]
-[n(A∩B) + n(B ∩C) + n(A∩C)].

Proof, n (A U B U C)

= n[(AUB)UC]

= n (A U B) + n (C) - n [(A U B) ∩C]

= re (A U B) + n (C) - n [(A ∩ C) U (B ∩ C)]

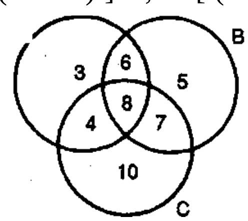
= n(A) + n(B)-n(A∩B)+n(C)

$-n(A \cap C) + n(B \cap C) - n(A \cap C \cap B \cap C)] = [n(A) + n(B) + n(C) + n(A \cap B \cap C)]$
$-[n(A \cap B) + n(A \cap C) + n(B \cap C)].$

Ex.5, In *a* group of athletic teams in a school, 21 are in the basket ball team; 26 in the hockey team and 29 in the football team. If 14 play hockey and basket ball; 12 play football and basket ball; 15 play hockey and football and 8 play all the three games.

Find: (i) how many players are there in all; *(ii)* how many play football only.

Sol. Let A, B, C be the sets of players forming basket ball team; hockey team and football team respectively. Then, $n(A) = 21$; $n(B) = 26$; $n(C) = 29$;
$n(A \cap B) = 14$; $n(A \cap C) = 12$; $n(B \cap C) = 15$ & $n(A \cap B \cap C) = 8$.
$\therefore n(A \cup B \cup C) = [n(A) + n(B) + n(C) + n(A \cap B \cap C)] - [n(A \cap B) + n(A \cap C) + n(B \cap C)]$ A, $= [(21 + 26 + 29 + 8) - (14 + 12 + 15)] = 43.$



Thus, there are 43 players in the group. With the given data, we may draw the Venn diagram as shown herewith.

Clearly, the number of players playing football only

$= [29 - (7 + 8 + 4)] = 10.$

Ex.6. A class has *175* students. The following is the description showing the number of students studying one or more of the following subjects in this class.

Mathematics 100; Physics 70; Chemistry 46; Mathematics and Physics 30; Mathematics and Chemistry 28; Physics and Chemistry 23; Mathematics, Physics and Chemistry 18.

Find (i) How many students are enrolled in Mathematics alone, Physics alone and Chemistry alone ;

(ii) the number of students who have not offered any of these three subjects.

Sol. Let A, B, *C* denote the sets of students enrolled in Mathematics, Physics and Chemistry respectively.

Let us denote the number of elements contained in each bounded region by small letters a, b, *c*, d , e, f, g as shown in_the figure.

Using the given data, we have

$$a + b + c + d = 100;$$
$$b + c + e + f = 70;$$
$$c + d + f + g = 46;$$
$$b + c = 30;$$
$$c + d = 28;$$
$$c + f = 23;$$
$$c = 18.$$

Solving these equations, we get:

c = 18; f = 5 ; d = 10 ; b = 12 ; g = 13 ; e = 35 & a = 60.

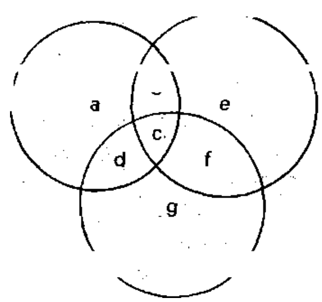
(i) Number of students enrolled in :

Mathematics alone = a = 60; *Physics alone* = e = 35; *Chemistry alone* = g = 13.

(ii) Number of students who have not offered any of these three subjects = $[175 - (a + b + c + d + e + f + g)]$

$= [175 - (60 + 12 + 18 + 10 + 35 + 5 + 13)]$

$= (175 - 153) = 22.$



Ex.7. If A and B be two sets containing 3 and 6 distinct elements respectively, what can be the minimum number of elements in $A \cup B$? Find also, the maximum number of elements in $A \cup B$.

Sol. Since n (A U B) = n (A) + *n* (B) - n (A ∩ B), it is clear that n (A U B) is maximum or minimum according as n (A ∩*B)* is minimum or maximum respectively.

Case I: When n (A ∩B) is minimum, i.e. n (A ∩ B) = 0.

This happens when A ∩B = ϕ.

In this case, *n* (A U B) = n (A) + n (B) = (3 + 6) = 9.

∴Maximum number of elements in A ∪ B is 9.

Case II: When n (A ∩ B) is maximum, i.e. n (A ∩B) = 3.

In this case, n (A U B) = *n* (A) + *n* (B) - n (A ∩ B) = (3 + 6 - 3) = 6.

.-. Minimum number of elements in A u *B* **i**s 6.

2.6 CARTESIAN PRODUCT OF TWO SETS If A and B are two non-empty sets, then the set of all ordered pairs (a, b) such that a ∈ A and b ∈ B, is called the cartesian product of A and B, to be denoted by A x B.

Thus, A x B = {(a, b): ***a*** ∈ A and b ∈B }.

If A = ϕ or B = ϕ we define A x B = ϕ

Remarks, (i) If A and B are finite sets, then $n(A \times B) = n(A).n(B)$.

(ii) If either A or B is an infinite set, then A x B is an infinite set.

Ex.1. If A = {1, 3, 5 } and B = { 2, 3 }, find $A \times B$ and $B \times A$. Show that $A \times B \neq B \times A$

Sol. A x B = {(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)}.

And, B x A = {(2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5)}.

Thus, $A \times B \neq B \times A$.

Ex.2. If A = {1, 2, 3), B = (3, 4} and C = {4, 5, 6} find

(i) $A \times (B \cup C)$ (ii) $A \times \{B \cap C)$

(iii) $(A \times B) \cap B \times C)$.

Sol. (i) $B \cup C = \{3,4\} \cup \{4, 5,6\} = \{3,4, 5,6\}$.

$\therefore A \times (B \cup C) = \{1,2,3\} \times \{3,4,5,6\} = \{(1, 3), (1, 4), (1, 5), (1, 6),$
$(2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$.

(ii) $B \cap C = \{3,4\} \cap \{4,5,6\} = \{4\}$.

$\therefore A \times (B \cap C) = \{1, 2, 3\} \times \{4\}$

$= \{(1, 4), (2,4) (3,4)1.$

(iii) $A \times B = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$

and $B \times C = \{(3,4), (3, 5), (3,6), (4, 4), (4, 5), (4, 6)\}$

$\therefore (A x B) \cap (B x C) = \{(3,4)\}$.

Ex.3. *If* $A \times B = \{ (3, 2), (3, 4), (5, 2), (5, 4)\}$, find A and B.

Sol. Clearly, A is the set of all first co-ordinates of elements of A x B, while B is the set of all second co-ordinates of elements of A x B.

$\therefore A = \{3,5\}$ and $B = \{2,4\}$.

Ex.4. A and B are two sets given in such a way that A x B contains 6 elements. If three elements of A x B be (1, 3), (2, 5) and (3, 3), find its remaining elements.

Sol. Since (1, 3), (2, 5) & (3, 3) are in A x B, it follows that 1, 2, 3 are in A and 3, 5 are in B**.**

$\therefore A \times B = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 3), (3, 5)\}$.

Hence, the remaining elements of A x B are:

(1,5), (2, 3) and (3, 5).

2.7 APPLICATION OF SETS TO BUSINESS MATHEMATICS

Set theory is seen as the foundation from which virtually all of mathematics can be derived. For example, structures in abstract algebra, such as groups, fields and rings, are setsclosed under one or more operations. One of the main applications of naive set theory is constructing relations. A relation from a domain *A* to a co-domain *B* is a subset of the Cartesian product A × B. Given this concept, we are quick to see that the set *F* of all ordered pairs (x, x^2), where *x* is real, is quite familiar. It has a domain set **R** and a codomain set that is also R, because the set of all squares is subset of the set of all reals. If placed in functional notation, this relation becomes $f(x) = x^2$. The reason these two are equivalent is for any given value, *y* that the function is defined for, its corresponding ordered pair, (y, y^2) is a member of the set *F*.

Theorem1: For any three sets A, B, C prove that

Ax(B u C) = (AxB)U(AxC);

Proof. Let (a, b) be an arbitrary element of A x (BUC).

Then, (a, b) ∈ A x (B U C)

=> a ∈A & b ∈(B ∪ C)

=>a∈ A & (b ∈ B or b∈ C)

=> (a ∈A & b ∈B) or (a ∈A & b ∈C)

=> (a, b) ∈ A x B or (a, b) ∈ (A x C)

=> (a, b) ∈ (A x B) U (A x C).

∴ A x (B U C) ⊆ (A x B) U (A x C).

Similarly, (A x B) U (A x C) ⊆ A x (B ∪ C).

Hence, A x (B UC) = {AxB) ***U***(AxC).

Theorem 2. For any three sets A, B, C prove that Ax(BUC) = (AxB)U(AxC).

Proof. Let (a, b) be an arbitrary element of A x (B UC).

Then, (a,b)&Ax(BUC)

a ∈ A and ***b*** ∈ (B U C)

=> a ∈A and (b ∈B & b ∈C)

=> (a ∈ A & b ∈ B) and (a ∈ A & D∈C)

=> (a, b) e (Ax B) and (a,b)∈(AxC)

=> (a, ***b)*** ∈ (A x B) U (A x C).

∴Ax(B-C)⊆(AxB)(U(Ax C).

Similarly, $(A \times B) \cup (A \times B) \subseteq A \times (B \cup C)$.

Hence, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Theorem 3. For any sets A, B, C, D prove that: $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof. Let $(a, b) \in (A \times B) \cap (C \times D)$.

Then, $(a,b) \in (A \times B) \cap (C \times D)$

$\Rightarrow (a, b) \in (A \times B)$ and $(a, b) \in (C \times D)$

$\Rightarrow (a \in A \ \& \ b \in B)$ and $(a \in C \ \& \ b \in D)$

$\Rightarrow (a \in A \ \& \ a \in C)$ and $(b \in B \ \& \ b \in D)$

$\Rightarrow a \in (A \cap C)$ and $b \in (B \cap D)$.

$\Longrightarrow (a,b) \in (A \cap C) \times (B \cap D)$.

$\therefore$.: $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$.

Similarly, $(A \cap C) \times (B \cap D) \in (A \times B) \cap (C \times D)$.

Hence, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Theorem 4. If $A \subseteq B$ & $C \subseteq D$, prove that $A \times C \subseteq B \times D$.

Proof. Let $A \subseteq B$, $C \subseteq D$ and let $(a, c) \in A \times C$.

Then, $(a, c) \in A \times C \Rightarrow a \in A \ \& \ c \in C$

$\Rightarrow a \in B \ \& \ c \in D \ [\because A \subseteq B \ \& \ C \subseteq D]$

$\Longrightarrow (a, c) \in B \times D$.

$\therefore A \times C \subseteq B \times D$.

Thus, $A \subseteq B \ \& \ C \subseteq D \Longrightarrow (A \times C) \Longrightarrow (B \times D)$.

Theorem 5. For non-empty sets A and B prove that $A \times B = B \times A \Leftrightarrow A - B$.

Proof. Let $A = B$. Then, $A \times B = A \times A$ and $B \times A = A \times A$.

And, therefore, in this case, $A \times B = B \times A$.

Conversely, let $A \times B = B \times A$ and let $\boldsymbol{x} \in A$.

Then, $x \in A \Rightarrow (a, b) \in A \times B$ for $b \in B$

(x, b) e B x A [∵ A x B = B x A]

=> x ∈B.

∴A⊆B. Similarly, B⊆A.

Consequently, A = B.

Theorem 6. If A ⊆ B, show that (AxA) ⊆(Ax B) ∩ (B x A).

Proof. Let A ⊆B and let (a, b) ∈ A x A.

Then, (a,b)∈ AxA=>(a∈ A & b∈ A)

=> (a ∈ A, 6 ∈ A) & (a ∈ A, ***b*** ∈ A)

[Repetition of statements]

=f (a ∈ A, ***b*** ∈ B) & (a ∈B, 6 ∈A) [∵ A ⊆B]

=> (a,6)∈(AxB)& (a, b)∈ (B xA)

=> (a,b) ∈ (AxB)∩(BxA).

This shows that AxA⊆(AxB)⊆(BxA).

Hence, A⊆B => A x A⊆(A x B) ∩ (BxA).

Theorem 7. If A and B are two non-empty sets having n elements in common, then prove that A x B and B x A have n^2 elements in common.

Proof. Let C **=** (A ∩ B). Then, we claim that

$C \times C$ = (AxB)∩(BxA),

since (a, 6) ∈ C x C

<=> a ∈C and ***b*** ∈ C

⇔a ∈A∩B and 6 ∈A ∈B [∵C = A∩B]

⇔ (a ∈ A & a ∈ B) and (6 ∈ A & f∈€ B)

<=> (a ∈ A & 6 ∈ B) and (a ∈B & ***b*** ∈ A)

<=> (a, fe) e A x B and (a, 6) e B x A

<=> (a, fe) e (AxB)n(Bx A). .

∴C x C = (AxB)∩(BxA).

And, since C x C has re^2 elements, so (A x B) n (B x A) has n^2 elements, *i.e.* (A x B) and (B x A) have elements in common.

Ex.7. A and B are two sets having two elements in common. If n (A) = 5 and n (B) = 3, find n(AxB) and n ((AxB)∩(BxA)}.

Sol. n{AxB) = n(A).n (B) = (5 x 3) = 15.

Since A and B have 2 elements in common, it follows that A x B and B x A have 2^2 i.e. 4 elements in common.Hence, n {(A x B)∩ (B x A)} = 4.

2.8 SUMMARY

- A set is not necessarily a collection of similar objects or a relationship between these objects. 'Set' thus differs from the everyday use of the term 'collection'. It is inconceivable that a normal mind will put in the same category a book, a star, the number seven, and so on. Collections are usually built to conform to a certain criterion. This constraint expresses a principle of our normal, conceptual way of thinking. It is an extremely useful and a very important principle of logical thinking. A 'collection' that groups totally unrelated objects or phenomena is intuitively and behaviourally nonsense.
- In set theory, one uses the term 'empty set'. Intuitively and practically, an empty set is nonsense. Mathematical generalisation has led to the creation of this concept for the sake of consistency in set theory. Therefore, it makes sense to affirm that the 'common points' of two parallel lines constitute an empty set. In everyday language, we would say that 'the two lines have no common points'.
- A set containing a single element (a singleton) is also, intuitively, contradictory. A set ('a collection') must have, in everyday terminology, several elements. A 'collection' containing only one element is, practically, unacceptable. A single element cannot constitute a set, a collection. It is only by mathematical generalization that one element is also an instance of a set.

- The algebra of sets defines the properties and laws of sets, the set-theoretic operations of union, intersection,and complementation and the relations of set equality and set inclusion. It also provides systematic procedures for evaluating expressions, and performing calculations, involving these operations and relations.
- The algebra of sets is the set-theoretic analogue of the algebra of numbers. Just as arithmetic addition and multiplication are associative and commutative, so are set union and intersection; just as the arithmetic relation "less than or equal" is reflexive, anti symmetric and transitive, so is the set relation of "subset".
- It is the algebra of the set-theoretic operations of union, intersection and complementation, and the relations of equality and inclusion. For a basic introduction to sets see the article on sets, for a fuller account see naive set theory, and for a full rigorous axiomatic treatment see axiomatic set theory.

2.9 SELF ASSESSMENT EXERCISES

1. Define set, subset.
2. Give the purpose of drawing Venn diagrams.
3. If $A = \{1, 3, 5\}$, $B = \{3, 5, 6\}$ and $C = \{1, 3, 7\}$

   (i) Verify that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

   (ii) Verify $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4. Let $A = \{a, b, d, e\}$, $B = \{b, c, e, f\}$ and $C = \{d, e, f, g\}$

   (i) Verify $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   (ii) Verify $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

5. If A = { 2, 3, 5, 7 } and B = (2, 4, 6 }, find (i) A x B (ii) B x A (iii) A x A (iv) BxB.

6. If each one of the two sets A and B has 3 elements, how many elements are there in A x B ? If three of the elements in A x B be (1, 2), (3, 3) and (5, 5), find its remaining elements.

7. A market survey on 1000 consumers shows that 720 consumers liked product A and 450 liked product B. What is the least number that must have liked both the products ?

8. In a class of 50 boys; 35 like horror movies; 30 like war movies and 5 like neither. Find the number of those that like both. Hint, n (A) = 35, n (B) - 30 and n(AuB)= 45.

9. In a committee, 50 people speak Hindi, 20 speak Bengali and 10 speak both Hindi and Bengali. How many speak at least one of these two languages ?

10. In a group of persons, each one knows either Hindi or Tamil. If 100 persons know Hindi, 50 know Tamil and 25 know both; how many persons are there in all, in the group ?

11. In a certain locality of Delhi, there are 1000 families. A survey indicated that 300 subscribe to The Hindustan Times daily newspaper and 250 subscribe to The Statesman daily newspaper. Of these two categories, 100 subscribe to both. Find the number of families which do not subscribe to any of these newspapers.

12. In a class of 25 students, 12 have taken mathematics, 8 have taken mathematics but not biology. Find the number of students who have taken mathematics and biology and those who have taken biology but not mathematics.

13. In a class, 18 students offered physics; 23 offered chemistry and 24 offered mathematics. Of these, 13 are in both chemistry and-mathematics; 12 in physics and chemistry; 11 in mathematics & physics and 6 in all the three subjects. Find :

(i) how many students are there in the class;

(ii) how many offered mathematics but not chemistry;

(iii)how many are taking exactly one of the three subjects.

14. If A = { 1, 2, 3 }, B = { 2, 3, 4 }, C = { 1, 3, 4) and D = { 2, 4, 5 }, then verify that:

$$(A x B) \cap (C x D) = (A \cap C) x (B \cap D).$$

15. Let A = {a, b, c, d, B = {6, *c*, *e*] and C = [a, e]. Find:

(i) $A U B$ (ii) $A U G$ (iii) $B U C$

(iv) $A \cap B$ (v) $A \cap C$ (vi) $B U C$

16. Let A = {1, 2, 4, 5), B = {2, 3, 5, 6 } *and C* = {4, 5, 6, 7). Verify the following identities.

(i) $(A U B) U C = A U (B U C);$

(ii) $(A \cap B) \cap C = A \cap (B \cap C);$

(iii) $A U (B \cap C) = (A U B) \cap (A U C);$

(iv) $A \cap (B U C) = (A \cap B) U (A \cap C).$

17. Let A = {a, 6, c, d, e }, B = {a, c, e, g } and C = { b, e, f, g). Verify the following identities :

(i) $A \cap (B - C) = (A B \cap) - (A \cap C);$

(ii) $A - (B U C) = (A - B) \cap (A - C);$

(iii) $A - (B \cap C) = (A - B) U (A - C).$

18. Let A = {1, 2, 3, 4 }, B = {2, 3, 5, 7 } and C = { 1, 3, 5, 6 }. Verify that $A \cap (B \Delta C) = (A \cap B) A (A \cap C).$

19. Let A = {1, 4, 7, 8 }, B = { 4, 6, 8, 9 } and C = { 3, 4, 5, 7 } be three subsets of a universal set, U = { 1, 2, 3, 4, 5, 6, 7, 8, 9), find:

(i) $A \cap (B - C)$ (ii) $A - (B - C)$ (iii) $A U B'$ (iv) $A' \cap (B' - C)$

(v) $A U (B \cap C)$ (vi) $A \Delta C.$

20. Using laws of operations on sets, prove that :

(i) $A \cap n\ (A' \cup B) = A \cap B$ (ii) $A \cap (A \cup B)' = 6$

(iii) $A - (A-B) = A \cap B$

(iv) $(A \cup B) - (A \cap B) = (A-B) \cup (B-A)$.

(v) (v) $(A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)'$.

21. Let A = {a, b, c }, B = { b, c, d, e } and C = {c, d, e, f} be three subsets of the universal set U = [a, b, c, d, e, f } .

Verify the following identities :

(i) $(A \cap B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

(iii) $A - B = A \cap B'$ (iv) $A \cap (B - C) = (A \cap B) - (A \cap C)$.

22. Let $A = \{ x \in N : x < l)$, $B = (2,3,5,7\}$ and C = {1, 3, 5, 7, 9 }. Show that: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

23. For any sets A and B, prove that: $A' - B' = B - A$.

24. Prove that :

(i) $A \subseteq B \Rightarrow A \cup C \subseteq B \cup C$;

(ii) $A \subseteq B \Rightarrow A \cap C c B \cap C$;

(iii) $(A-B) \cap (B-A) = o$;

(iv) $A \cap (B-C) = (A \cap B) - C$.

25. Prove that: $A \cap B = 6 \Rightarrow A \subseteq B'$.

26. Prove that: $A \cap B = A \cup C$ and $A \cap B = A \cap C \Rightarrow B = C$,

27. Find whether the following statements are true or false : .1

(i) $A - (B \cap C) = (A-B) \cap (A - C)$;

(ii) $(A \cup B)' = A' \cup B'$;

(iii) $A \cap B = \phi \Rightarrow A = \phi$ or $B = 6$;

2.10 SUGGESTED READINGS

- Mathematics: R.S Aggarwal, Bharati Bhawan, Patna.
- A Text book of Business Mathematics: Dr. A.K. Arte.
- Business Mathematics. Navaneethan. P.

STRUCTURE

3.1 INTRODUCTION

Succession of numbers of which one number is designated as the first, other as the second, and another as the third and so on gives rise to what is called a sequence. Sequences have wide applications. In this unit we shall discuss particular types of sequences called arithmetic sequence, geometric sequence and will also establish the relation between A.P and G.P.

A sequence is a collection of numbers specified in a definite order by some assigned law, whereby a definite number an of the set can be associated with the corresponding positive integer *n*.

The different notations used for a sequence are.

1. $a_1, a_2, a_3, \ldots, a_n, \ldots.$
2. a_n , $n = 1, 2, 3, \ldots$
3. $\{a_n\}$

Let us consider the following sequences:

1.) 1, 2, 4, 8, 16, 32 ...

2.) 1, 4, 9, 16, 25...

3.) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots \ldots \ldots \ldots ..$ etc.

In the above examples, the expression for nth term of the sequences are as given below:

(1) $a_n = 2^{n-1}$ (2) $a_n = n^2$ (3) $a_n = \frac{n}{n+1}$ (4) $a_n = 1/n$

For all positive integer n.

Also for the first problem in the introduction, the terms can be obtained from the relation

$a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2}$

A finite sequence has a finite number of terms. An infinite sequence contains an infinite number of terms.

Remark 1: It is not necessary that every sequence may be defined by some explicit formula for the nth term. But, there must be a certain rule for finding the succeeding terms of the sequence. e.g. 2, 3, 5, 7, 11, 13 ,...... is an infinite sequence of all prime numbers in which there is no explicit formula for finding the general term.

Series: By adding the terms of a sequence, we obtain a series. A series is finite or infinite according as the number of terms added is finite or infinite.

Progressions: Sequences following certain patterns are called progressions. We will discuss about two types of progressions, namely arithmetical and geometrical.

3.2 OBJECTIVES:

After studying this unit, you will be able to:

- Describe the concept of a sequence (progression)
- Define an A.P.
- Find common difference and general term of a A.P
- Calculate the common difference or any other term of the A.P. calculate the fourth quantity of an A.P. given three of S, n, a and d.
- State that a geometric progression is a sequence increasing or decreasing.
- Identify G.P.'s from a given set of progressions.
- Find the common ratio and general term of a G.P.
- Calculate the common ratio and any term when two of the terms of the G.P. are given.
- Write progression when the general term is given.

3.3 ARITHMETIC PROGRESSION

It is a sequence in which each term, except the first one, differs from its preceding term by a constant, called the common difference.

In an A.P. we usually denote the first term by a, the common difference by d and the nth term by a_n Clearly, $d = (a_n - a_{n-1})$.
A finite sequence of numbers with this property is called an arithmetic progression. A sequence of numbers with finite terms in which the difference between two consecutive terms is the same non-zero number is called the Arithmetic Progression or simply A. P. The difference between two consecutive terms is called the common difference of the A. P. and is denoted by 'd'.

In general, an A. P. whose first term is a and common difference is d is written as $a, a + d, a + 2d, a + 3d,$

Also we use a_n to denote the nth term of the progression.

3.4 FINDING THE NTH TERM

Let us consider A. P. $a, a + d, a + 2d, a + 3d,$

Here, first term $(a1) = a$

Second term (a2) = a + d = a + (2 – 1) d,

Third term (a3) = a + 2d = a + (3 – 1) d

By observing the above pattern, nth term can be written as:

$a_n = a + (n – 1) d$

Hence, if the first term and the common difference of an A. P. are known then any term of

A. P. can be determined by the above formula.

Sometimes, the nth term of an A. P. is expressed, in terms of n, e.g. $t_n = 2_{n-1}$.

In that case, the A. P. will be obtained by substituting $n = 1, 2, 3,$ in the expression.

In this case, the terms of the A. P. are 1, 3, 5, 7, 9,......

Ex.1. 5, 8, 11, 14, 17,... is an A.P. whose first term is 5 and the common difference is (8 - 5) = 3.

Ex.2. 5, 2, - 1, - 4, - 7 ... is an A.P. whose .first term is 5 and the common difference is (2 - 5) = - 3.

Ex.3. Show that the progression 7, 2, -3,-8,... is on A.P. Find its 16th term and the general term.

Sol. Since (2- 7) = (- 3 - 2) = [- 8 - (- 3)] = - 5, it follows that the difference between any two consecutive terms is constant. So, it is an A.P.

Its first term = 7 and common difference = - 5.

$a_{n = a + (n-1)d}$

$$\therefore an = 7 + (n - 1) \times -5$$

= 7+ 5-5n

= 12-5n

16th term = 7 + (16 - 1) × (- 5) = - 68.

Ex.4. Which term of the progression 4, 9, 14, 19,... is 109 ?

Sol. Clearly, the given progression is an A.P.

in which the first term = 4 and the common difference = 5.
Let its nth term be 109.
Then, 4 + (re - 1) x 5 = 109 or n = 22.
Hence, the 22nd term of the given A.P. is 109.

Ex.5. How many terms are there in the A.P. 7, 13, 19, 25, ... 205 ?

Sol. Suppose the given A.P. contains n terms.
Then, clearly the nth term is 205.
∴ 7 + (n - 1) x 6 = 205 => n = 34.

Thus, the given A.P. contains 34 terms.

Ex.6. Is 301 a term of the sequence 5, 11, 17, 23,...?

Sol. The given sequence is clearly an A.P. with first term = 5 and common difference = 6.
Let the nth term of the given sequence be 301.
Then, 5 + (n - 1) x 6 = 301 => n = 50 if N=50
∴ 301 is not a term of the given sequence.

Ex.7. Find the second term and the nth term of an A.P. whose 6th term is 12 and the 8th term is 22.

Sol. Let 'a' be the first term and d, the common difference of the given A.P.

Then, 6th term = a + 5d and 8th term = a + 7d.

$\therefore$ $a + 5d = 12$ ...(i) and $a + 7d = 22$ ...(ii)

Solving (i) and (ii), we get $a = -13$ and $d = 5$.

$\therefore$ Second term $= (a + d) = (-13 + 5) = -8$.

And, nth term $= [a + (n - 1) \times d] = [-13 + (n - 1) \times 5] = (5n - 18)$.

Ex.8. Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.

Sol. Let the numbers be $(a-3d)$, $(a-d)$, $(a + d)$ and $(a + 3d)$.

Then,

$(a-3d) + (a-d) + (a+d) + (a + 3d) = 20 \Rightarrow 4a = 20$ i.e. $a = 5$.

And, $(a-3d)^2 + (a-d)^2 + (a+d)^2 + (a + 3d)^2 = 120$

$\Rightarrow 4(a^2 + 5d^2) = 120$

$\Rightarrow (25 + 5d^2) = 30$ $[\because a = 5]$

$\Rightarrow d^2 = 1$

$\Rightarrow d = \pm 1$.

So, the numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

Ex.9. Find three numbers in A.P. whose sum is 15 and the product is 80.

Sol. Let the numbers be $(a - d)$, a, $(a + d)$.

Then,

$(a - d) + a + (a + d) = 15 \Rightarrow 3a = 15$ $a = 5$.

And, $(a-d).a.(a + d) = 80 \Rightarrow a(a^2 - d^2) = 80$

$= 5(25-d^2) = 80$ $[\therefore a = 5]$

$= d^2 = 9 \Rightarrow d = \pm 3$.

So, the required numbers are 2, 5, 8 or 8, 5, 2.

Ex.10. Find the 19th term from the end of the A.P. 2, 6, 10, 14,..., 86.

Sol. 19th term from the end $= [L-(n-1)d] = [86 - (19 - 1) \times 4] = 14$.

Ex. 11: The common difference of an A. P. is 3 and the 15th term is 37. Find the first term.

Sol: Here, $d = 3$, $a_{15} = 37$, and $n = 15$

Let the first term be a. We have

$a_n = a + (n - 1)\, d$

$37 = a + (15 - 1)\, 3$

or, $37 = a + 42$

$a = -5$

Thus, first term of the given A. P. is – 5.

3.4.1 Finding nth term from the last term of an A.P.

Let ***a*** be the first term, d the common difference and *l* the last term of a given A.P. Then, the A.P. is a, (a + d), (a + 2d),... , (l - 2d), (l - d), l.

∴ Last term = L= L-(1- l)d;

2nd term from the end = (l - d) = l - (2 -1) d;

3rd term from the end = (l- 2d) = l - (3 - 1) d;

∴ nth term from the end = L- (n - 1) d.

Ex.. Find the 19th term from the end of the A.P. 2, 6, 10, 14, ..., 86

Sol. 19th term from the end = [l-(n-l)d] = [86 - (19 - 1) x 4] = 14.

3.5 SUM OF 'N' TERMS OF AN A.P

Let us consider an A.P. containing n terms, in which the first term is a, the common difference is d and the last term is *l*.

Let S_n be the sum of these terms.

Then,

$S_n = a + (a + d) + (a + 2d) + \dots + (l - 2d) + (l - d) + l$ ...(i)

Writing the above series in a reverse order, we get

$S_n = \mathbf{l + (t - d) + (l - 2d) + \dots + \{a + 2d) + (a + d) + a}$ ...(ii)

Adding the corresponding terms of (i) & (ii), we get

$2S_n = [(a + l) + (a + l) + \dots\dots n \text{ times}] = n(a + l)$

$\therefore S_n = n/2(a + l)$.

But, the last term, **l** = nth term = $a + (n - 1)\, d$.

$\therefore S_n = n/2\,(a + a + (n - l)\; d] = n/2[2a + (n - l)d]$.

Ex.1. Find the sum of 17 terms of the A.P. 5, 9, 13, 17,....

Sol. Clearly, $a = 5$; $d = 4$ & $n = 17$.

$\therefore$ Sum of 17 terms $= n/2[2a + (n-1) d]$

$$= 17/2 \times [2 \times 5 + (17 - 1) \times 4] = 629.$$

Ex.2. Find the sum of the series: $2 + 5 + 8 + ... + 182$.

Sol. Let there be n terms in the given series.

Then, $a_n = 182 \Rightarrow [2 + (n - 1) \times 3] = 182 \quad n = 61$.

$\therefore$ Required Sum $= n/2\ (a + 1) = 61/2 \times (2 + 182) = 5612$.

Ex.3. How many terms of A.P. 18, 15, 12,... are needed to give the sum 45 ? Explain the double answer.

Sol. Let the sum of n terms be 45.

Then, $n/2\ [2 \times 18 + (n-1) \times (-3)] = 45$

$\Rightarrow 3n^2 - 39n + 90 = 0$

$\Rightarrow n^2 - 13n + 30 = 0$

$\Rightarrow (n - 3)(n - 10) = 0$

$\Rightarrow n = 3$ or $n = 10$.

$\therefore$ Sum of 3 terms = Sum of 10 terms = 45

Here, the common difference is negative.

Out of the terms from 4th to 10th, some are positive and some are negative and their sum is zero.

Thus, the sum of 3 terms as well as that of 10 terms is 45.

Ex.4. Find the sum of first 35 terms of an A.P., in which $a_2 = 2$ and $a_7 = 22$.

Sol. Let a be the first term and d the common difference.

Then, $a + d = 2$ and $a + 6d = 22$.

Solving these equations, we get $a = -2$ and $d = 4$.

$S_{35} = 35/2 \times [2 \times (-2) + (35-1) \times 4] = 2310.$

Ex.5. Find the sum of all integers between 100 and 1000, which are divisible by7.

Sol. The integers between 100 and 1000, divisible by 7 are 105, 112, 119, ..., 994.

$\therefore$ Required sum = (105 + 112 + 119 + ... + 994).

This is an arithmetic series with first term 105 and common difference 7.

Let n be the number of terms in th is series.

Then, $\mathbf{a_n}$ = 994 => 105 + (n - 1) x 7 = 994 =>.n = 128.

$\therefore$ Required sum = S_{128} = 128/2 x (105 + 994) = 70336.

Ex.6. Solve: 1 + 6 + 11 + 16 + ... + n = 148.

Sol. Clearly,the numbers forming the given series are in A.P.

with ***a*** = 1 and ***d*** = 5.

Let there be n terms in this series.

Then, S_n= [2a + (n - 1) d] = 148

=[2 x 1 + (n - 1) x 5] = 148

=> $5n^2 - 3n - 296 = 0$

=> (n - 8) (5n + 37) = 0

=> n = 8 [***neglecting the negative value of n]***

= nth term =8thterm = [l + (8-l) x5] = 36.

Ex.7. If the sum of n terms of a progression be a quadratic expression in n, show that it is an A.P.

Sol. Let $\mathbf{S_n} = an^2 + bn + c$, where ***a,b,c*** are constants & $a \neq 0$.

Then, $S_{n-1} = a(n-1)^2 + b(n-1) + c$.

$\therefore a_n = (S_n - S_{n-1})$

$= (an^2 + bn + c) - [a(n-1)^2 + b(n-1) + c]$

$= a(2n-1) + b$

= 2an + (b- a), which is a linear expression in n

But, whenever, the nth term of a progression is a linear expression in ***n***, then it is an A.P.

Hence the given progression is an A.P.

Ex.8. *The first, second and the last terms of an A.P. are a, b, c respectively. Show that the sum of the A.P. is*

$$\frac{(b+c-2a)(a+c)}{2(b-a)}.$$

Sol. Let the number of terms in the given A.P. be n.

Then, first term $= a$; common difference $= (b-a)$

and the last term, $c = a + (n-1)(b-a)$.

Now, $c = a + (n-1)(b-a) \;\Rightarrow\; n = \dfrac{(b+c-2a)}{(b-a)}$.

$\therefore$ sum of the A. P. $= \left(\dfrac{n}{2}\right)$ [first term + last term]

$$= \frac{(b+c-2a)}{2(b-a)} \cdot (a+c) = \frac{(b+c-2a)(a+c)}{2(b-a)}.$$

Ex.9. *The sum of the first p, q, r terms of an A.P. are a, b, c respectively. Show that*

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0.$$

Sol. Let x be the first term and d the common difference. Then,

$$S_p = a \Rightarrow \left(\frac{p}{2}\right)\cdot[2x+(p-1)d] = a$$

$$\Rightarrow \frac{a}{p} = x+(p-1)\cdot\frac{d}{2}$$

$$\Rightarrow \frac{a(q-r)}{p} = x(q-r)+(p-1)(q-r)\cdot\frac{d}{2} \qquad \text{...(i)}$$

Similarly,

$$S_q = b \Rightarrow \frac{b(r-p)}{q} = x(r-p)+(q-1)(r-p)\cdot\frac{d}{2} \qquad \text{...(ii)}$$

And $S_r = c \Rightarrow \frac{c(p-q)}{r} = x(p-q)+(r-1)(p-q)\cdot\frac{d}{2}$ ...(iii)

Adding (i), (ii) and (iii), we get

Ex.9. The sum of n terms of an A.P. **is** $(2n + 3n^2)$. Determine the A.P. and find its nth term.

Sol. Let $S_n = (2n + 3n^2)$.

Then, $S_{n-1} = [2(n-1) + 3(n-1)^2] = (3n^2-4n + 1)$.

$\therefore a_n = (S_n-S_{n-1}) = (2n + 3n^2)-(3n^2-4n + 1) = (6n-1)$.

Thus, $a_1 = (6\times1-1) = 5$ & $a_2 = (6\times2-1) = 11$

Clearly, a = 5 and d = (11 - 5) = 6.

$\therefore$The A.P. is 5, 11, 17, 23,...

Also, $a_n = (6n-1) \Rightarrow a_r=(6r-1)$.

Ex.10. If the first term, of an A.P. is 2 and the sum of first five terms is equal to one-fourth of the sum of the next five terms, find the sum of first 30 terms.

Sol. We have

$(t_1 + t_2 + t_3 + t_4 + t_5) = \frac{1}{4}(t_6 + t_7 + t_8 + t_9 + t_{10})$

$= 1/4 [(t_1 + t_2 + \ldots + t_{10}) - (t_1 + t_2 + \ldots + t_5)]$

or $S_5 = 1/4(S_{10} - S_5)$

$\Rightarrow 5\,S_5 = S_{10}$

$\Rightarrow 5 \times 5/2[2 \times 2 + (5-1)d] = 10/2\,[2 \times 2 + (10-1)d]$

$\Rightarrow 50(1 + d) = 20 + 45d$

$\Rightarrow d = -6.$

$\therefore S_{30} = 30/2 \times [2 \times 2 + (30-1) \times (-6)] = -2550.$ ......

Ex. 11: The 35th term of an A. P. is 69. Find the sum of its 69 terms.

Sol. Let a be the first term and d be the common difference of the A. P.

We have,

$a_{35} = a + (35 - 1)\,d = a + 34\,d.$

$a + 34\,d = 69$ ... (i)

Now by the formula, $S_n = n/2(2a + (n-1)d)$

We have

$S_{69} = 69/2\ \underline{[2a + (69 - 1)d\ ]}$

$= [2 + (69 - 1)d\ ]$

$= 69\,(a + 34d)$ [using (i)]

$= 69 \times 69 = 4761.$

Ex.12. ***If the sum of m terms of an A.P. be n and the sum of n terms be m, show that the sum of $(m + n)$ terms is $-(m + n)$.***

Sol. Let a be the first term and d the common difference.

Then, $S_m = n \Rightarrow \left(\dfrac{m}{2}\right) \cdot [\,2a + (m-1)\,d\,] = n$

$$\Rightarrow 2am + m\,(m-1)\,d = 2n \qquad \text{...(i)}$$

And, $S_n = m \Rightarrow \left(\dfrac{n}{2}\right) \cdot [\,2a + (n-1)\,d\,] = m$

$$\Rightarrow 2an + n\,(n-1)\,d = 2m \qquad \text{...(ii)}$$

Subtracting (ii) from (i), we get:

$$2a(m-n)+\{(m^2-n^2)-(m-n)\}\cdot d=(n-m)$$

or $2a(m-n)+(m-n)(m+n-1)\,d=2(n-m)$

or $2a+(m+n-1)\,d=-2$ ...(iii)

$$\therefore\quad S_{m+n}=\left(\frac{m+n}{2}\right)\cdot[2a+(m+n-1)d]$$

$$=\left(\frac{m+n}{2}\right)\cdot(-2)=(m+n)\cdot\quad[\textit{Using (iii)}]$$

Ex.13. *If the sum of p terms of an A.P. is the same as the sum of its q terms, show that the sum of its (p + q) terms is zero.*

Sol. Let a be the first term and d the common difference. Then,

$$S_p=S_q\ \Rightarrow\ \left(\frac{p}{2}\right)\cdot[2a+(p-1)d]=\left(\frac{q}{2}\right)\cdot[2a+(q-1)d]$$

$$\Rightarrow\ (1-p-q)d=2a \qquad ...(i)$$

$$\therefore\quad S_{p+q}=\left(\frac{p+q}{2}\right)\cdot[2a+(p+q-1)d]$$

$$=\left(\frac{p+q}{2}\right)\cdot[\,(1-p-q)d+(p+q-1)d\,]\quad[\textit{Using (i)}]$$

$$=0.$$

Ex.14. *If S_1, S_2, S_3 be the sum of n, $2n$, $3n$ terms respectively of an A.P., prove that $S_3 = 3(S_2 - S_1)$.*

Sol. Let a be the first term and d the common difference. Then,

$$3(S_2 - S_1) = 3\left\{\left(\frac{2n}{2}\right)\cdot[2a + (2n-1)d] - \left(\frac{n}{2}\right)\cdot[2a + (n-1)d]\right\}$$
$$= \left(\frac{3n}{2}\right)[2a + (3n-1)d]$$
$$= S_3.$$

Ex.15. *The ratio between the sum of n terms of two arithmetical progressions is $(7n + 1) : (4n + 27)$. Find the ratio of their 11th terms.*

Sol. Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the given APs. Then, their sums of n terms are given by:

$$S_n = \frac{n}{2}[2a_1 + (n-1)d_1] \qquad ...(i)$$

Ex.15 (a) How many 2 - digit positive integers are divisible – by 4?

Sol. The smallest 2-digit positive integer divisible by 4 is 12.The largest 2- digit positive integer divisible by 4 is 96.

All the 2 digit positive integers are terms of an Arithmatic progression with 12 being the first from and 96 being last term.

The common difference is 4

The nth term, an =a_1 +(n - 1)d, where a, is the first form , ‘n’ number of terms and ‘d’ the common difference.

So, 96 =12 + (n -1) x 4 Or 84 = (n-1) x 4

Or (n-1) =21 : there are 22.

Hence , n = 22 2- digit integers that are divisible by 4.

and $S'_n = \frac{n}{2}\ [\ 2a_2 + (n-1)\ d_2\ ]$ ...(ii)

On dividing, we get :

$$\frac{2a_1 + (n-1)\ d_1}{2a_2 + (n-1)\ d_2} = \frac{(7n+1)}{(4n+27)} \qquad \text{...(iii)}$$

$\therefore$ *The ratio of their 11th terms*

$$= \frac{a_1 + 10d_1}{a_2 + 10d_2} = \frac{2a_1 + 20d_1}{2a_2 + 20d_2}$$

$$= \frac{2a_1 + (21-1)\ d_1}{2a_2 + (21-1)\ d_2} = \frac{(7 \times 21 + 1)}{(4 \times 21 + 27)} = \frac{148}{111} \quad [\ \textit{Putting } n = 21 \textit{ in } \textbf{(iii)}\ ]$$

Hence, the required ratio is 148 : 111.

Ex.16. *The ratio of the sums of m and n terms of an A.P. is $m^2 : n^2$. Show that the ratio of mth and nth terms is $(2m-1) : (2n-1)$.*

Sol. Let a be the first term and d the common difference. Then,

$$\frac{S_m}{S_n} = \frac{m^2}{n^2} \Rightarrow \frac{\left(\frac{m}{2}\right) .\ [\ 2a + (m-1)\ d\ ]}{\left(\frac{n}{2}\right) .\ [\ 2a + (n-1)\ d\ ]} = \frac{m^2}{n^2}$$

$$\Rightarrow \frac{2a + (m-1)\ d}{2a + (n-1)\ d} = \frac{m}{n}$$

$$\Rightarrow 2an + n\ (m-1)\ d = 2am + m\ (n-1)\ d$$

$$\Rightarrow 2a\ (n-m) = (n-m)\ d$$

$$\Rightarrow d = 2a.$$

$$\therefore \quad \frac{m\text{th term}}{n\text{th term}} = \frac{a + (m-1)\ d}{a + (n-1)\ d} = \frac{(2m-1)}{(2n-1)} . \quad [\ \because\ d = 2a\ ]$$

Ex.17. *If the mth term of an A.P. is (1/n) and the nth term is (1/m), show that the sum of mn terms is $\frac{1}{2}(mn+1)$.*

Sol. Let a be the first term and d the common difference.

Then, $a + (m-1)\ d = \frac{1}{n}$ ...(i) $\left[\ \because\ t_m = \frac{1}{n}\ \right]$

And, $a + (n-1)\ d = \frac{1}{m}$ ...(ii) $\left[\ \because\ t_n = \frac{1}{m}\ \right]$

Subtracting (ii) from (i) we get

$(m-n)\ d = \left(\frac{1}{n} - \frac{1}{m}\right) = \left(\frac{m-n}{mn}\right)$ and therefore, $d = \frac{1}{mn}$.

Putting $d = \frac{1}{mn}$ in (i), we get

$$a + \frac{(m-1)}{mn} = \frac{1}{n} \text{ or } a = \left[\frac{1}{n} - \frac{(m-1)}{mn}\right] = \frac{1}{mn}.$$

$$\therefore \quad S_{mn} = \left(\frac{mn}{2}\right).[\,2a + (mn-1)\,d\,]$$

$$= \left(\frac{mn}{2}\right)\cdot\left[\frac{2}{mn} + \frac{(mn-1)}{mn}\right]$$

$$= \frac{1}{2}(mn+1).$$

Ex.18. ***A man repays a loan of Rs. 3250 by paying Rs. 20 in the first month and then increases the payment by Rs. 15 every month. How long will it take him to clear the loan ?***

Sol. Let it be cleared in n months.

Clearly, the amounts form an A.P. with first term 20 and the common difference 15.

$$\therefore \quad \left(\frac{n}{2}\right).[\,2 \times 20 + (n-1) \times 15\,] = 3250$$

$$\Rightarrow 3n^2 + 5n - 1300 = 0$$

$$\Rightarrow (n-20)(3n+65) = 0$$

$$\Rightarrow n = 20 \text{ or } n = \frac{-65}{3}.$$

Neglecting the fractional value, we get $n = 20$.

3.6 REPRESENTATION OF AN AP

Theorem 1. If to each term of an A.P. a fixed non-zero number is added, then the resulting progression is also an A.P.

Proof. Consider an A.P. a, a + d, a + 2d, a + 3d.....

Let be a non-zero number positive or negative which is added to each term of the above A.P.

The resulting progression is

(a + k), (a+d + k), (a + 2d + k), (a + 3d + k).....

Clearly, the difference between two consecutive terms of this progression is constant and equal to d.

Thus, the new progression is an A.P. with first term a + k and the common difference d.

Theorem 2. If each term of a given A.P. is multiplied or divided by a given non-zero fixed number k, then the resulting progression is an A.P.

Proof. Let the given A.P. be a, a + d, a + 2d, a + 3d,...

Let k be a non-zero number.

Then, the progression obtained by multiplying each term of the given A.P. by k is

ak, (ak + dk), (ak + 2dk), (ak + 3dk),.......

which is clearly an A.P. with first term ak and the common difference dk.

Again, on dividing each term of the given A.P. by k, we obtain the progression

a/k, (a+d)/k, (a+2d)/k, (a+3d)/k, ……..

which is clearly an A.P. with first term A/K and the common difference (d/k).

Ex.1. If a - b, c are in AP., show that

I. (b + c), (c + a) and (a + 6) are in A.P.

II. a^2 (b + c), b^2 (c + a) and c^2 (a + 6) are in A.P.

Sol. Since a, b, c are in A.P., we have 26 = (a + c)……. ...(i)

I. (6 + c), (c + a), (a + 6) will be in A.P.

if (c + a) - (6 + c) = (a + 6) - (c + a)

i.e. if a - b = b- c

i.e. if 2b = a + c, which is true by (i).

∴ a, 6, c are in A.P.

=> (b + c), (c + a), (a + 6) are in A.P.

II. a^2 (b + c), b^2 (c + a), c^2 (a + b) will be in A.P.

if b^2 (c + a)-a^2 (b +c) = c^2 (a + b)-b^2(c + a)

i.e. if c (b^2 - a^2) + ab (b - a) = a (c^2 - b^2) + bc (c - b)

i.e. if (b- a) (ab + bc + ca)'= (c - b) (ab + bc +ca) i.e. if (b - a) = (c - b)

i.e. if 2b = (a + c),

which is true by (i).

$\therefore$ a, b, c are in A.P.

=> $a^2(b + c)$, $b^2(c + a)$, $c^2(a + b)$ are in A.P.

Ex.2. *If a, b, c are in A.P., show that*

$$\frac{1}{(\sqrt{b}+\sqrt{c})}, \frac{1}{(\sqrt{c}+\sqrt{a})}, \frac{1}{(\sqrt{a}+\sqrt{b})} \text{ are in A.P.}$$

Sol. Since a, b, c are in A.P., we have $2b = (a + c)$ **...(i)**

Now, $\frac{1}{(\sqrt{b}+\sqrt{c})}, \frac{1}{(\sqrt{c}+\sqrt{a})}, \frac{1}{(\sqrt{a}+\sqrt{b})}$ will be in A.P.

if $\frac{1}{(\sqrt{c}+\sqrt{a})} - \frac{1}{(\sqrt{b}+\sqrt{c})} = \frac{1}{(\sqrt{a}+\sqrt{b})} - \frac{1}{(\sqrt{c}+\sqrt{a})}$

i.e. if $\frac{(\sqrt{b}-\sqrt{a})}{(\sqrt{c}+\sqrt{a})(\sqrt{b}+\sqrt{c})} = \frac{(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

i.e. if $\frac{(\sqrt{b}-\sqrt{a})}{(\sqrt{b}+\sqrt{c})} = \frac{(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})}$

i.e. if $b - a = c - b$

i.e. if $2b = a + c$, which is true by **(i).**

$\therefore$ a, b, c are in A.P. $\Rightarrow \frac{1}{(\sqrt{b}+\sqrt{c})}, \frac{1}{(\sqrt{c}+\sqrt{a})}, \frac{1}{(\sqrt{a}+\sqrt{b})}$ are in A.P.

Ex.3. *If a, b, c, are in A.P., show that*
$[(b+c)^2 - a^2]$, $[(c+a)^2 - b^2]$, $[(a+b)^2 - c^2]$ are in A.P.

Sol. Since a, b, c are in A.P., we have $2b = (a+c)$ **...(i)**

Now $[(b+c)^2 - a^2]$, $[(c+a)^2 - b^2]$, $[(a+b)^2 - c^2]$ will be in A.P.

if $[(c+a)^2 - b^2] - [(b+c)^2 - a^2] = [(a+b)^2 - c^2] - [(c+a)^2 - b^2]$

i.e. if $(a+b+c)(c+a-b) - (a+b+c)(b+c-a)$
$$= (a+b+c)(a+b-c) - (a+b+c)(c+a-b)$$

i.e. if $(a+b+c)[(c+a-b) - (b+c-a)]$
$$= (a+b+c)[(a+b-c) - (c+a-b)]$$

i.e. if $2(a-b) = 2(b-c)$

i.e. if $(a-b) = (b-c)$

i.e. if $2b = (a+c)$, which is true by **(i)**.

$\therefore$ a, b, c are in A.P.

$\Rightarrow [(b+c)^2 - a^2], [(c+a)^2 - b^2], [(a+b)^2 - c^2]$ are in A.P.

Ex.4. *If a, b, c are in A.P. show that :*

(i) $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ *are in A.P.*

(ii) $\dfrac{a(b+c)}{bc}, \dfrac{b(c+a)}{ca}, \dfrac{c(a+b)}{ab}$ *are in A.P.*

Sol. (i) Since a, b, c are in A.P., the terms obtained by dividing each term of this A.P. by abc are also in A.P.

Consequently, $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are in A.P.

(ii) a, b, c are in A.P.

$\Rightarrow \dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ are in A.P. [*dividing each term by abc*]

$\Rightarrow \left(\dfrac{ab+bc+ca}{bc}\right), \left(\dfrac{ab+bc+ca}{ca}\right), \left(\dfrac{ab+bc+ca}{ab}\right)$ are in A.P.

[*multiplying each term by* $(ab+bc+ca)$]

$\Rightarrow \left(\dfrac{ab+bc+ca}{bc} - 1\right), \left(\dfrac{ab+bc+ca}{ca} - 1\right), \left(\dfrac{ab+bc+ca}{ab} - 1\right)$

are in A.P. [*Adding* -1 *to each term*]

Ex. 4(a) Find the number of terms in the geometric progression 6,12,24,………..1536.

Sol. $a_1 = 6$ $a_2 = 12$, $a_3 = 24$, $a_n = 1536$

$\therefore$ $r = a_2 / a_1 = 12/6 = 2$

an = 1536

$ar^{n-1} = 1536 \Rightarrow 6x(2)^{n-1}$

$\Rightarrow 1536/6 = 2^{n-1}$ $\Rightarrow$ $2^8 = 2^{n-1}$

$\Rightarrow 8 = {}^{n-1}$ $\Rightarrow$ $8+1 = n$

$\Rightarrow 9 = n$

Hence , 1536 is the 9th term

$\therefore$ Number of terms in the above G.P. is 9.

$$\Rightarrow \frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab} \text{ are in A.P.}$$

Ex.5. If a^2, b^2, c^2 are in A.P., show that

$$\frac{1}{(b+c)}, \frac{1}{(c+a)}, \frac{1}{(a+b)} \textit{ are in A.P.}$$

Sol. If a^2, b^2, c^2 are in A.P.

$\Rightarrow [\, a^2 + (ab+bc+ca)\,], [\, b^2 + (ab+bc+ca)\,], [\, c^2 + (ab+bc+ca)\,]$

are in A.P. [*Adding* $(ab+bc+ca)$ *to each term*]

$\Rightarrow (a+b)(c+a), (a+b)(b+c), (c+a)(b+c)$ are in A.P.

$$\Rightarrow \frac{1}{(b+c)}, \frac{1}{(c+a)}, \frac{1}{(a+b)} \text{ are in A.P.}$$

Ex.6. *If* $\frac{(b+c-a)}{a} \cdot \frac{(c+a-b)}{b}, \frac{(a+b-c)}{c}$ *are in A.P. prove that* $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ *are also in A.P.*

Sol. $\frac{(b+c-a)}{a}, \frac{(c+a-b)}{b}, \frac{(a+b-c)}{c}$ are in A.P.

$$\Rightarrow \left\{\frac{(b+c-a)}{a}+2\right\}, \left\{\frac{(c+a-b)}{b}+2\right\}, \left\{\frac{(a+b-c)}{c}+2\right\}$$

are in A.P. [*adding* 2 *to each term*]

$$\Rightarrow \frac{(a+b+c)}{a}, \frac{(a+b+c)}{b}, \frac{(a+b+c)}{c} \text{ are in A.P.}$$

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. [*dividing each term by* $(a+b+c)$]

Ex.7. *If* $(b-c)^2, (c-a)^2, (a-b)^2$ *are in A.P., prove that*

$$\frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)} \textit{ are in A.P.}$$

Sol. $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P.

$\Rightarrow (c-a)^2 - (b-c)^2 = (a-b)^2 - (c-a)^2$

$\Rightarrow (b-a)(2c-a-b) = (c-b)(2a-b-c)$ ...(i)

Now, $\frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)}$ will be in A.P.

if $\frac{1}{(c-a)} - \frac{1}{(b-c)} = \frac{1}{(a-b)} - \frac{1}{(c-a)}$

i.e. if $\frac{(a+b-2c)}{(c-a)(b-c)} = \frac{(b+c-2a)}{(a-b)(c-a)}$

i.e. if $(a-b)(a+b-2c) = (b-c)(b+c-2a)$

i.e. if $(b-a)(2c-a-b) = (c-b)(2a-b-c)$, which is true by (i).

Thus, if $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P.,

then $\frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)}$ are in A.P.

Ex.8. *If* $(a^2+2bc), (b^2+2ac), (c^2+2ab)$ *are in A.P. show that*

$$\frac{1}{b-c}, \frac{1}{(c-a)}, \frac{1}{(a-b)} \quad \text{are in A.P.}$$

Sol. $a^2+2bc, b^2+2ac, c^2+2ab$ are in A.P.

$\Rightarrow (a^2+bc-ab-ac), (b^2+ac-ab-bc), (c^2+ab-bc-ac)$

are in A.P. [*on subtracting* $(ab+bc+ac)$ *from each term*]

$\Rightarrow (a-b)(a-c), (b-c)(b-a), (c-b)(c-a)$ are in A.P.

$\Rightarrow \frac{-1}{(b-c)}, \frac{-1}{(c-a)}, \frac{-1}{(a-b)}$ are in A.P.

[*dividing each term by* $(a-b)(b-c)(c-a)$]

$\Rightarrow \frac{1}{(b-c)}, \frac{1}{(c-a)}, \frac{1}{(a-b)}$ are in A.P.

[*multiplying each term by* -1]

Ex.9. *If a, b, c are in A.P., prove that*

$$a^3+4b^3+c^3 = 3b\,(a^2+c^2).$$

Sol. Since a, b, c are in A.P., we have $b = \frac{1}{2}(a+c)$.

$\therefore$ R.H.S. $= 3 \cdot \frac{1}{2}(a+c)(a^2+c^2) = \frac{3}{2}(a^3+c^3+ac^2+a^2c)$;

L.H.S. $= a^3 + 4 . \left(\frac{a+c}{2}\right)^3 + c^3 = \frac{3}{2}(a^3+c^3+ac^2+a^2c)$.

$\therefore$ L.H.S. = R.H.S.

3.7 GEOMETRIC PROGRESSION (GP)

A sequence of numbers in which every term, except the first one, bears a constant ratio with its preceding term, is called a geometrical progression, abbreviated as G.P.

The constant ratio is called the common ratio of the G.P.

Ex.1. 2, 6, 18, 54, 162,... is a G.P., since 6/2=18/6=54/18=162/54=3 which is constant.

This is a G.P. in which first term is ***2*** and the common ratio is ***3.***

Ex.2. 64, - 16, 4, - 1,….... is a G.P. with first term 64 and common ratio = -16/64= -1/4.

Therefore, A sequence of numbers in which the ratio of any term to the term which immediately precedes is the same non zero number (other than1), is called a geometric progression or simply G. P. This ratio is called the common ratio.

The most general form of a G. P. with the first term a and common ratio r is $a, ar, ar^2, ar^3, \ldots$

Let us consider a geometric progression with the first term a and common ratio r.

Then its terms are given by

$a, ar, ar^2, ar^3, \ldots$

In this case, $t_1 = a = ar^{1-1}$

$t_2 = ar = ar^{2-1}$

$t_3 = ar^2 = ar^{3-1}$

$t_4 = ar^3 = ar^{4-1}$

3.7.1 Some properties of a G. P.

(i) If all the terms of a G. P. are multiplied by the same non-zero quantity, the resulting series

is also in G. P. The resulting G. P. has the same common ratio as the original one.

If a, b, c, d,... are in G. P.

then $ak, bk, ck, dk \ldots$ are also in G. P.

(ii) If all the terms of a G. P. are raised to the same power, the resulting series is also in G. P.

Let *a, b, c, d* ... are in G. P.

the *ak, bk, ck, dk,* ... are also in G. P.

The common ratio of the resulting G. P. will be obtained by raising the same power to the

original common ratio.

3.8 FINDING THE nth TERM OF GP

Let ***a*** be the first term and ***r*** the common ratio of a given G.P.

Then, First term, $t_1 = a = ar^{(1-1)}$

Second term, $t_2 = ar = ar^{(2-1)}$

Third term, $t_3 = ar^2 = ar^{(3-1)}$

nth term, $t_n = ar^{(n-1)}$

Ex.1. Find the 9th term and the general term of the progression 3, 6, 12, 24,…

Sol: The given progression is clearly a G.P. with first term, a = 3 and common ratio, r = 2.

$\therefore$ 9thterm, $a_9 = ar^{(9-1)} = ar^8$

$= (3 \times 2^8) = (3 \times 256) = 768.$

Also, general term, $a_n = ar^{(n-1)} = 3 \times 2^{(n-1)}$

Ex.2. Which term of the G.P. 5, 10, 20, 40…..... is 5120 ?

Sol. Here a = 5 and r = 2.

Let the nth term be 5120.

Then, $a_n = 5120 \Longrightarrow 5 x 2^{(n-1)} = 5120$

$= 2^{(n-1)} = 1024 = 2^{10}$

$\Rightarrow (n - 1) = 10$

$\Rightarrow n = 11.$

Ex.3. If the 4th and 9th terms of a G.P. be 54 and 13122 respectively, find the G.P.

Sol. Let ***a*** be the first term and ***r*** the common ratio.

Then, $\mathbf{a_4} = 54 \Rightarrow ar^3 = 54$………….. ...(i)

And, $a_9 = 13122 \Rightarrow ar^8 = 13122$……. ...(ii)

On dividing (ii) by (i), we get:

Ar8/ar3 = 13122/54= 243=>.r^5= 243=3^5=> r = 3

Putting r = 3 in (i), we get a = 2.

Hence, the required G.P. is 2, 6, 18, 54, ...

Ex.4. The third term of a G.P. is 4**.** Find the product of its first five terms.

Sol. Let a be the first term and r the common ratio.

Then, $a3 = ar^2 = 4$.

$\therefore\ t_1, t_2, t_3, t_4, t_5 = a\{ ar) (ar^2) (ar^3) (ar^4)$

$= a^5 r^{10} = (ar^2)^5 = 4^5 = 1024 \quad [\therefore ar^2 = 4]$

Ex.5. The 4th, 7th and l0tn terms of a G.P. are a, b, c respectively.

Show that $b^2 = ac$.

Sol. Let A be the first term and R the common ratio.

Then, $t_4 = a$; $t_7 = b$ and $t_{10} = c$

$\Rightarrow AR^3 = a;\ AR^6 = b$ and$AR^9 = c$

$\Rightarrow b^2 = (AR^6)^2 = A^2 R^{12} = (AR^3)(AR^9) = ac.$

Ex.6. If a, b, c are in G.P., prove that log a, log b, log c are in A.P.

Sol. a, b, c are in G.P.

=>$b^2 = ac$

=> 2 log b = log ***a*** + log c [Taking Logarithms]

=> log a, log 6, log c are in A.P.

Ex.7. If a, b, c are in G.P. and $a^{1/x} = b^{1/y} = c^{1/c}$, then show that x, y, z are in A.P.
Sol. Let $a^{1/x} = b^{1/x} = c^{1/x} = k$.
Then, $a = k^x$, $b = k^y$ and $c = k^{z.}$
Now, a, b, c are in G.P.
$\Rightarrow b^2 = ac$
$\Rightarrow (k^y)^2 = k^x . k^z = k^{x+z}$
$\Rightarrow k^{2y} = k^{x+z}$
$\Rightarrow 2y = (x+z)$.
This shows that x, y, z are in A.P.

Ex. 8. If a, b, c, d are in G.P., show that

$$(b - c)^2 + (c - a)^2 + (d - bf = (a r- d)^2.$$

Sol. Let r be the common ratio of the G.P. a, b, c, d.
Then, $b = ar$; $c = ar^2$ and $d = ar^3$.

$\therefore$ L.H.S. $= (6 - c)^2 + (c - a)^2 + (d - b)^2$
$= (ar - ar^2)^2 + (ar^2 - a)^2 - (ar^3 - ar)^2$
$= a^2 r^2 (1 - r)^2 + a^2 (r^2 - If + a^2 r^2 (r^2 - l)^2$
$= a^2 (/-^6 - 2r^3 + l) = a^2 (l - r^3)^2$
$= (a - ar^3)^2 = (a - d)^2 =$ R.H.S.

3.8.1 Finding the nth term from the End of a G.P

Let 'a' be the first term, **r** the common ratio and **I** the last term of a given G.P.

Then, The second term from the end = $l/r = l/r^{(2-1)}$
The third term from the end = $l/r^2 = l/r^{(3-1)}$

$\therefore$ nth term from the end = $\frac{l}{r(n-1)}$

Ex.1. Find the 5th term from the end of the G.P.

3, 6, 12, 24,..., 12288.

Sol. 5th term from the end = $l/r^4 = 12288/2^4 = 768$

Ex.2. Show that in a G.P., the product of the two terms equidistant from the beginning and the end is equal to the product of the first and the last term.

Sol. Let $a . ar . ar^2 \ldots\ldots\ldots, 1$ be the given G.P. r

Then, the nth term from the beginning is given by

$$a_n = ar^{n-1}$$

Also, the nth term from the end is given by

$t'_n = l/r^{n-1}$ where l is the last term of the G.P.

$\therefore \boldsymbol{an . a'n = (arn-1 \times \frac{l}{rn} - 1) = al}$ = (first term) x (last term).

Ex.3. Find all sequences which are simultaneously A.P. and G.P.

Sol. Let (a_n) be a sequence which is both A.P. and G.P.

Then, a_n, a_{n+1}, a_{n+2} being the three consecutive terms of the A.P.,

we have $a_{n+1} = a_n + a_{n+2}/2$, $n > \mathbf{1}$. ...(i)

Now, let r be the common ratio of the G.P. $\{a_n\}$, then

$$a_n = a_1 r^{n-1};\ a_{n+1} = a_1 r^n \text{ and } a_{n+2} = a_1 r^{n+1}.$$

Putting these values in (i), we get

$a_1 r^n = a_1 r^{n-1} + a_1 r^{n+1}/2$

or $r^2 - 2r + l = 0$ or $(r-l)^2 = 0$ or $r = l$.

$\therefore a_1 = a_2 = a_3 = \ldots$ i.e. the constant sequence is the only sequence which is both an A.P. and a G.P.

Ex.4. The sum of three numbers which are consecutive terms of an A.P. is 21. If the second number is reduced by 1 and the third is increased by 1, we obtain three consecutive terms of a G.P. Find the numbers.

Sol. Let the numbers be $a-d,\ a, a+d$.

Then, $a - d + a + a + d = 21$

=> $3a = 21$

i.e. $a = 7$.

So, the numbers are $7 - d, 7, 7 + d$.

$\therefore 7-d,\ 7-1,\ 7+d\ +1$

i.e. $7-d,\ 6,\ 8+d$ are in G.P.

$\Rightarrow 6/7\text{-D} = 8\text{+D}/6$

$\Rightarrow d^2 + d - 20 = 0$

$\Rightarrow (d + 5)(d-4) = 0$

$\Rightarrow d = -5$ or $d = 4$.

$\therefore$The numbers are 12, 7, 2 or 3,7,11.

Ex.5. Find four numbers in G.P. in which the third term is greater than the first by 9 and the second term is greater than the fourth by 18.

Sol. Let the required numbers be a, ar, ar^2, ar^3.

Then, $a_3 - a_1 = 9$ and $a_2 - a_4 = 18$

$\Rightarrow ar^2 - a = 9$ and $a r - a r^3 = 18$

$\Rightarrow a(r^2 - l) = 9$ and $a r(1 - r^2) = 18$

$ar(l-r^2)/a(r^2-1) = 18/9$ [on dividing]

$\Rightarrow r = -2$

Putting $r = -2$ in $a (r^2 - 1) = 9$, we get $a = 3$.

$\therefore$ The numbers are 3, -6, 12 & - 24.

3.8.2 Problems on G.P

For solving problems on G.P. it is always convenient to take

(i) three numbers in G.P. as (a/r), a, ar ;

(ii) four numbers in G.P. as (a/r^3), (a/r), ar, ar^3 ;

(iii) five numbers in G.P. as (a/r^2), (a/r), a, ar, ar^2 ;

(iv) the numbers as $a, ar, ar^2, \ldots$, when their product is not given.

Ex.1. The sum of three numbers in G.P. is 35 and their product is 1000. Find the numbers.

Sol. Let the numbers be a/r, a, ar.

Then, a/r, a, ar= 1000 => a^3 = 1000 => a =10.

Also, a/r + a + ar = 35 => (10/r +10+10r) = 35 [a=10]

=> $10(1 + r + r^2) = 35 r$

=> $2r^2 - 5r + 2 = 0$

=> (2 r -I) (r - 2) = 0

=> r = ½ or r = 2.

The required numbers are 20, 10, 5 or 5, 10, 20.

Ex.2. Find three numbers in G.P. whose sum is 13 and the sum of whose squares is 91.

Sol. Let the required numbers be a/r, a and ar.

Then, a/r + a + ar = 13 ....(i) r

And, $a^2/r^2 + a^2 + a^2r^2 = 91$…………………(ii)

On squaring (i), we get

$(a^2/r^2 + a^2 + a^2r^2) + 2(a^2/r^2 + a^2 + a^2r) = 169$

Or $(a^2/r^2 + a^2 + a^2r^2) + 2a(a/r + a + ar) = 169$

or 91 +26a =169 [Using (ii) *and* (i) **]**

This gives, a = 3.

Putting a = 3 in (i), we get

$3r^2 - 10r + 3 = 0$

1. (3r-l)(r-3) = 0

=> r = 1/3 or r = 3.

∴The numbers are 9, 3, 1 or 1, 3, 9.

Ex.3. Find three numbers in G.P. whose sum is 52 and the sum of whose products in pairs is 624.

Sol. Let the required numbers be a, ar, ar^2.

Then,

$$a + ar + ar^2 = 52 \Rightarrow a(1 + r + r^2) = 52 \ldots\ldots\ldots\ldots \text{(i)}$$

Also, $a \cdot ar + ar \cdot ar^2 + a \cdot ar^2 = 624 = a^2 r(1 + r + r^2) = 624$ ...(ii)

Dividing (ii) by (i), we get ar =12 or a = 12/r

Substituting a = 12/r in (i), we get

$$(12/r)(1 + r + r^2) = 52 \Rightarrow 3(1 + r + r^2) = 13r$$
$$\Rightarrow 3r^2 - 10r + 3 = 0$$
$$\Rightarrow (3r - 1)(r - 3) = 0$$
$$\Rightarrow r = 1/3 \text{ or } r = 3.$$

$\therefore$ a = 36 or 0 = 4.

Hence, the numbers are 36, 12, 4 or 4, 12, 36.

3.9 SUM OF 'N' TERMS OF GP

Let **a** be the first term and **r** the common ratio of a given G.P.

Let Sn denote the sum of first **n** terms of this G.P. Then,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \ldots\ldots\ldots \text{(i)}$$

$$\therefore \quad rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n \ldots\ldots\ldots\ldots \text{(ii)}$$

On subtracting (ii) from (i), we get:

$$(1-r)S_n = (a - ar^n) = a(1 - r^n)$$

$$\therefore Sn = \frac{a(1 - rn)}{(1 - r)} = \frac{a(rn - 1)}{(r - 1)}$$

Now, if **l** be the last term of the G.P., then $l = ar^{n-1}$

$$\therefore Sn = \frac{(a - arn)}{(1 - r)} = \frac{(a - lr)}{(1 - r)} = \frac{(lr - a)}{(r - 1)}$$

Ex.1. *Find the sum of the following geometric series :*

(i) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ *to 12 terms ;*

(ii) $243 + 324 + 432 + \ldots$ *to n terms.*

Sol. (i) Here $a = 1$ and $r=\frac{1}{2}<1.$

$$\therefore \quad S_{12}=\frac{a(1-r^{12})}{(1-r)}=\frac{1\times\left[1-\left(\frac{1}{2}\right)^{12}\right]}{\left(1-\frac{1}{2}\right)}=\left(\frac{2^{12}-1}{2^{11}}\right)$$

$$=\left(\frac{4095}{2048}\right).$$

(ii) Here $a = 243$ and $r=\frac{324}{243}=\frac{4}{3}>1.$

$$\therefore \quad S_n=\frac{a(r^n-1)}{(r-1)}=\frac{243\times\left[\left(\frac{4}{3}\right)^n-1\right]}{\left(\frac{4}{3}-1\right)}=3^{(6-n)}\cdot(4^n-3^n).$$

Ex.2. *Find the sum of the series* $2 + 6 + 18 + \ldots + 4374.$

Sol. The given series is a geometric series in which

$a = 2,\quad r = 3\quad$ and $l = 4374.$

$\therefore$ The required sum $=\frac{(lr-a)}{(r-1)}=\left(\frac{4374\times3-2}{3-1}\right)=6560.$

Ex.3. *How many terms of the geometric series*

$$1 + 4 + 16 + 64 + \ldots.$$

will make the sum 5461 ?

Sol. Let the required number of terms be n.

Now, the given series is a geometric series in which

$$a = 1,\ r = 4 \text{ and } S_n = 5461.$$

$$\therefore\quad S_n = \frac{a\,(r^n - 1)}{(r - 1)} \Rightarrow 5461 = \frac{1 \times (4^n - 1)}{(4 - 1)}$$

$$\Rightarrow 4^n = 16384 = 4^7$$

$$\Rightarrow n = 7.$$

Hence, the sum of 7 terms is 5461.

Ex.4. *In a geometric progression,* $\{a_n\}$, *if*

$$a_1 = 3,\ a_n = 96 \text{ and } S_n = 189,$$

find n.

Sol. Here $a = 3$, $l = 96$ and $S_n = 189$.

$$\therefore\quad S_n = \frac{l\,r - a}{r - 1} \Rightarrow 189 = \frac{96\,r - 3}{(r - 1)} \Rightarrow r = 2.$$

Now, $l = ar^{n-1} \Rightarrow 96 = 3 \times 2^{n-1}$

$$\Rightarrow 2^{n-1} = 32 = 2^5$$

$$\Rightarrow n - 1 = 5, \text{ i.e. } n = 6.$$

Ex.5. *Sum the following series :*

(i) $5 + 55 + 555 + \ldots\ldots$ *to n terms ;*

(ii) $.4 + .44 + .444 + \ldots\ldots$ *to n terms.*

Sol. We have :

(i) $5 + 55 + 555 + \ldots..$ *to n terms*

$$= 5 \times [\,1 + 11 + 111 + \ldots\ldots \text{ to } n \text{ terms}\,]$$

$$= \frac{5}{9}\,[\,9 + 99 + 999 + \ldots.. \text{ to } n \text{ terms}\,]$$

$$= \frac{5}{9} \times [\,(10 - 1) + (10^2 - 1) + (10^3 - 1) + \ldots.. \text{ to } n \text{ terms}\,]$$

$$= \frac{5}{9} \times [\,(10 + 10^2 + 10^3 + \ldots\ldots. \text{ to } n \text{ terms}) - n\,]$$

$$= \frac{5}{9} \times \left[\,\frac{10 \times (10^n - 1)}{(10 - 1)} - n\,\right]$$

$$= \left(\frac{5}{81}\right) \times (10^{n+1} - 9n - 10)\,.$$

(ii) $.4 + .44 + .444 + \ldots\ldots$ to n terms

$$= 4 \times [\,.1 + .11 + .111 + \ldots\ldots \text{ to } n \text{ terms}\,]$$

$$= \frac{4}{9} \times [\,.9 + .99 + .999 + \ldots\ldots \text{ to } n \text{ terms}\,]$$

$$= \frac{4}{9} \times [\,(1 - .1) + (1 - .01) + (1 - .001) + \ldots\ldots \text{ to } n \text{ terms}\,]$$

$$= \frac{4}{9} \times [\,n - (\,.1 + .01 + .001 + \ldots\ldots \text{ to } n \text{ terms})\,]$$

$$= \frac{4}{9} \times \left(n - \frac{.1 \times [\,1 - (.1)^n\,]}{(1 - .1)} \right)$$

$$= \left(\frac{4}{81} \right) \times \left[9n - 1 + \frac{1}{10^n} \right].$$

Ex.6. *Sum to n terms the series whose nth term is* $(2^n + 3n)$.

Sol. We have $t_n = 2^n + 3n$.

$\therefore \quad t_1 = 2 + 3.1\,;$

$t_2 = 2^2 + 3.2\,;$

$t_3 = 2^3 + 3.3\,;$

... ...

$t_n = 2^n + 3.n.$

Adding columnwise, we get

$$S_n = (\,t_1 + t_2 + \ldots + t_n\,)$$

$$= (2 + 2^2 + \ldots + 2^n) + 3\,.\,(1 + 2 + \ldots\ldots + n)$$

$$= \left[\left(\frac{2^n\,.\,2 - 2}{2 - 1} \right) + 3\,.\,\frac{n\,(n+1)}{2} \right] = \left[\frac{4\,(2^n - 1) + 3n^2 + 3n}{2} \right].$$

Ex.7. *If S be the sum, P the product and R the sum of the reciprocals of n terms in a G.P., prove that* $P^2 = \left(\frac{S}{R} \right)^n$.

Sol. Let $a, ar, ar^2, ..., ar^{(n-1)}$ be the given G.P. Then,

$$S = \frac{a(1-r^n)}{(1-r)}; \qquad ...(i)$$

$$P = a \times ar \times ar^2 \times ... \times ar^{(n-1)} = a^n \, r^{[1+2+....+(n-1)]}$$

$$= a^n \, . \, r^{(n-1)n/2}$$

or $$P^2 = a^{2n} \, . \, r^{(n-1)n} \qquad ...(ii)$$

And, $$R = \left[\frac{1}{a} + \frac{1}{ar} + ... + \frac{1}{ar^{n-1}}\right] = \left(\frac{1}{a}\right) \cdot \frac{\left[\frac{1}{r^n} - 1\right]}{\left(\frac{1}{r} - 1\right)}$$

or $$R = \left(\frac{r}{a}\right) \cdot \frac{(1-r^n)}{(1-r) \, . \, r^n} = \frac{(1-r^n)}{a(1-r)\, r^{(n-1)}} \qquad ...(iii)$$

Dividing **(i)** by **(iii)**, we get

$$\frac{S}{R} = \frac{a(1-r^n)}{(1-r)} \cdot \frac{a(1-r)\, r^{(n-1)}}{(1-r^n)} = a^2 \, r^{n-1}$$

$$\therefore \left(\frac{S}{R}\right)^n = [\, a^2 \, r^{(n-1)} \,]^n = a^{2n} \, r^{(n-1)n} = P^2. \qquad [\textit{ Using } \textbf{(ii)} \,]$$

Ex.8. *If S_1, S_2 and S_3 be respectively the sum of n, $2n$ and $3n$ terms of a G.P., prove that :*

$$S_1 (S_3 - S_2) = (S_2 - S_1)^2.$$

Sol. Let a be the first term and r be the common ratio.

Then, $$S_1 (S_3 - S_2) = \frac{a(1-r^n)}{(1-r)} \cdot \left[\frac{a(1-r^{3n})}{(1-r)} - \frac{a(1-r^{2n})}{(1-r)}\right]$$

$$= \frac{a^2 \, r^{2n} (1-r^n)^2}{(1-r)^2}.$$

And, $$(S_2 - S_1)^2 = \left[\frac{a(1-r^{2n})}{(1-r)} - \frac{a(1-r^n)}{(1-r)}\right]^2 = \frac{a^2 \, r^{2n} (1-r^n)^2}{(1-r)^2}.$$

Hence, $S_1 (S_3 - S_2) = (S_2 - S_1)^2$.

3.10 SUM OF INFINITY

Let us consider a G.P. whose first term is ***a*** and the common ratio is ***r,*** where I ***r*** I < 1, ***i.e.*** the numerical value of r is less than 1.

Since I r I < 1, it follows that as the value of ***n*** increases, the value of r^n goes on decreasing. So, when ***n*** becomes indefinitely large, then r^n becomes indefinitely small.

i.e. when ***n*** approaches to infinity, then r^n approaches to 0.
And, we write $r^n \to 0$ as $n \Rightarrow \infty$.

Now. $S_n = a(1-r^n)/(1-r) = [a/(1-r) - ar^n/(1-r)]$

$\therefore \boldsymbol{S \propto = a/(1-r)}$

Ex.9. *Sum the following series to infinity :*

(i) $1-\frac{1}{3}+\frac{1}{3^2}-\frac{1}{3^3}+\ldots.\infty$

(ii) $(\sqrt{2}+1)+1+(\sqrt{2}-1)+\ldots\ldots\infty$

(iii) $\frac{2}{5}+\frac{3}{5^2}+\frac{2}{5^3}+\frac{3}{5^4}+\ldots\infty$

Sol. (i) Clearly, $a=1$ and $r=-\frac{1}{3}$.

$$\therefore \quad S_\infty=\frac{a}{(1-r)}=\frac{1}{\left[1-\left(-\frac{1}{3}\right)\right]}=\frac{1}{\left(1+\frac{1}{3}\right)}=\frac{3}{4}.$$

(ii) This is an infinite G.P. in which $a=(\sqrt{2}+1)$

and $r=\frac{1}{(\sqrt{2}+1)}\times\frac{(\sqrt{2}-1)}{(\sqrt{2}-1)}=(\sqrt{2}-1)$.

$$\therefore \quad S_\infty=\frac{a}{(1-r)}=\frac{(\sqrt{2}+1)}{[1-(\sqrt{2}-1)]}=\frac{(\sqrt{2}+1)}{\sqrt{2}(\sqrt{2}-1)}$$

$$=\frac{(\sqrt{2}+1)}{\sqrt{2}(\sqrt{2}-1)}\times\frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}=\left(\frac{3+2\sqrt{2}}{\sqrt{2}}\right)=\frac{(4+3\sqrt{2})}{2}.$$

(iii) *The Given Series*

$$=\left(\frac{2}{5}+\frac{2}{5^3}+\frac{2}{5^5}+\ldots\infty\right)+\left(\frac{3}{5^2}+\frac{3}{5^4}+\frac{3}{5^6}+\ldots\infty\right)$$

$$=\frac{\left(\frac{2}{5}\right)}{\left(1-\frac{1}{25}\right)}+\frac{\left(\frac{3}{25}\right)}{\left(1-\frac{1}{25}\right)}=\frac{13}{24}.$$

Ex.10. *Prove that* $6^{1/2}.6^{1/4}.6^{1/8}\ldots.\infty=6$.

Sol. $6^{1/2}.6^{1/4}.6^{1/8}\ldots\ldots\infty$

$$=6^{[(1/2)+(1/4)+(1/8)\ldots\ldots\infty]}$$

$$=6^1=6.\quad\left[\because\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots\infty\right)=\frac{(1/2)}{\left(1-\frac{1}{2}\right)}=1\right]$$

Ex.11. *If* $x=\left(a+\frac{a}{r}+\frac{a}{r^2}+\ldots\ldots\infty\right)$; $y=\left(b-\frac{b}{r}+\frac{b}{r^2}+\ldots.\infty\right)$

and $z=\left(c+\frac{c}{r^2}+\frac{c}{r^4}+\ldots\infty\right)$, *prove that* $\frac{xy}{z}=\frac{ab}{c}$.

Sol. On summing each infinite series, we get :

$$x = \frac{a}{\left(1-\frac{1}{r}\right)} = \frac{ar}{(r-1)},\ y = \frac{b}{\left(1+\frac{1}{r}\right)} = \frac{br}{(r+1)}\ \&\ z = \frac{c}{\left(1-\frac{1}{r^2}\right)} = \frac{cr^2}{(r^2-1)}.$$

$$\therefore\ \frac{xy}{z} = \left[\frac{ar}{(r-1)} \times \frac{br}{(r+1)} \times \frac{(r^2-1)}{cr^2}\right] = \left(\frac{ab}{c}\right).$$

Ex.12. If $x = 1 + a + a^2 + ... \infty$, *where* $|a| < 1$
and $y = 1 + b + b^2 + ... \infty$, *where* $|b| < 1$;
prove that $1 + ab + a^2 b^2 + ... \infty = \dfrac{xy}{(x+y-1)}$.

Sol. On summing each infinite geometric series, we get

$$x = \frac{1}{(1-a)} \text{ and } y = \frac{1}{(1-b)}.$$

$$\therefore\ \frac{xy}{(x+y-1)} = \frac{\frac{1}{(1-a)} \cdot \frac{1}{(1-b)}}{\left[\frac{1}{(1-a)} + \frac{1}{(1-b)} - 1\right]} = \frac{1}{(1-ab)}$$

$$= (1-ab)^{-1} = (1 + ab + a^2 b^2 + ... \infty).$$

Ex.13. If $y = x + x^2 + x^3 + ... \infty$, *prove that* $x = \dfrac{y}{(1+y)}$.

Sol. By summing the given infinite geometric series, we get

$$y = x\,(1 + x + x^2 + ... \infty) = x \cdot \frac{1}{(1-x)} = \frac{x}{(1-x)}.$$

Now, $\dfrac{x}{(1-x)} = y \Leftrightarrow x = y - xy \Leftrightarrow x\,(1+y) = y \Leftrightarrow x = \dfrac{y}{(1+y)}$.

Ex.14. *The sum of an infinite geometric series is* 8. *If its second term is* 2, *find the first term.*

Sol. Let a be the first term and r the common ratio.

Then $ar = 2$ **...(i)** and $\dfrac{a}{(1-r)} = 8$ **...(ii)**

Putting $r = \dfrac{2}{a}$ from **(i)** into **(ii),** we get

$\dfrac{a^2}{(a-2)} = 8$ or $a^2 - 8a + 16 = 0$ or $(a-4)^2 = 0$.

$\therefore$ $a = 4$, *i.e.* the first term of the G.P. is 4.

Ex.15. *Find an infinite G.P. whose first term is* 1 *and each term is the sum of all the terms which follow it.*

Sol. Let r be the common ratio.

Then, the G.P. is $1 + r + r^2 + ... \infty$.

$\therefore \quad 1 = r + r^2 + r^3 + ... \infty = \frac{r}{(1-r)}$

or $(1-r) = r$ or $2r = 1$ or $r = \frac{1}{2}$.

$\therefore$ Required G.P. is $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, ... \infty$.

Ex.16. *The sum of an infinite geometric series is* 15 *and the sum of the squares of these terms is* 45. *Find the series.*

Sol. Let a be the first term and r the common ratio.

Then, $(a + ar + ar^2 + ... \infty) = 15 \Rightarrow \frac{a}{(1-r)} = 15$ ...(i)

And, $(a^2 + a^2 r^2 + a^2 r^4 + ... \infty) = 45$

$$\Rightarrow a^2 (1 + r^2 + r^4 + ... \infty) = 45 \Rightarrow \frac{a^2}{(1-r^2)} = 45 \quad \text{...(ii)}$$

Squaring both sides of (i), we get $\frac{a^2}{(1-r)^2} = 225$ ...(iii)

On dividing (iii) by (ii) we get, $\frac{(1-r^2)}{(1-r)^2} = 5$

or $3r^2 - 5r + 2 = 0$ or $(r-1)(3r-2) = 0$

$\therefore \quad r = 1$ or $r = \frac{2}{3}$.

But, $r \neq 1$. So, $r = \frac{2}{3}$.

Putting $r = \frac{2}{3}$ in (i), we get $a = 5$.

$\therefore$ The required series is $\left(5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + ... \infty\right)$.

Ex.17. *If* $S_1, S_2, S_3, ..., S_p$ *denote the sums of infinite geometric series whose first terms are* 1, 2, 3, ... , p *respectively and whose common ratios are* $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{(p+1)}$ *respectively, show that :*

$$S_1 + S_2 + S_3 + ... + S_p = \frac{p\,(p+3)}{2}.$$

Sol. By the summation of given geometric series, we get

$$S_1=\frac{1}{\left(1-\frac{1}{2}\right)}=2\,;\ S_2=\frac{2}{\left(1-\frac{1}{3}\right)}=3\,;\ S_3=\frac{3}{\left(1-\frac{1}{4}\right)}=4\,;\,.....$$

and $\quad S_p=\dfrac{p}{\left[1-\dfrac{1}{(p+1)}\right]}=(p+1)$.

$$\therefore\ \ S_1+S_2+S_3+\ldots+S_p=2+3+4+\ldots+(p+1)$$

$$=\left(\frac{p}{2}\right).\,[\,2+(p+1)\,]=\frac{p\,(p+3)}{2}.$$

Ex.18. *Use geometric series to express* $0.555\ldots=0.\dot{5}$ *as a rational number.*

Sol. $0.\dot{5}=0.555\,....$

$$=0.5+0.05+0.005+\,.....\,\infty$$

[*An infinite geometric series with* $a=.5$ & $r=.1$]

$$=\frac{0.5}{(1-0.1)}=\frac{0.5}{0.9}=\frac{5}{9}.$$

Ex.19. Find the rational number whose decimal expansion is $0.1\dot{4}\dot{2}$.

Sol. $0.1\dot{4}\dot{2}=0.1424242\,....\,\infty$

$$=0.1+0.042+0.00042+0.0000042+\,....\,\infty$$

$$=\frac{1}{10}+\left[\frac{42}{10^3}+\frac{42}{10^5}+\frac{42}{10^7}+\,....\,\infty\right]$$

$$=\frac{1}{10}+\frac{\left(\dfrac{42}{10^3}\right)}{\left(1-\dfrac{1}{10^2}\right)}=\left(\frac{1}{10}+\frac{42}{990}\right)=\frac{141}{990}.$$

3.11 REPRESENTATION OF GP

Theorem 1. If all the terms of a G.P. be multiplied by the same non-zero number, then the new numbers form a G.P.

Proof. Let a, b, c, d, ... be **a** G.P. with common ratio r.

Then, b/a= c/b= d/c = ……= r =

Let ***k*** be a non-zero number.

Consider the progression ka, kb, kc, kd,...

Clearly, kb/ka = b/a = r; kc/kb = c/b r; kd/kc = d/c = r and so on.

This shows that ka, kb, kc, kd, ... form a G.P. with common ratio r.

Remark. If a, b, c, d .... are in G.P. and $\boldsymbol{k} \neq 0$, then

a/k,b/k,c/k,d/k, are in GP

Theorem 2. The reciprocals of terms in G.P. form a G.P.

Proof. Let a, b, c, d, ... be a G.P. with common ratio r.

Then, b/a = c/b = d/c=……r.

$$\therefore \frac{\frac{1}{b}}{\frac{1}{a}} = \frac{a}{b} = \frac{1}{r}; \frac{\frac{1}{c}}{} / \left(\frac{1}{b}\right) = \frac{b}{c} = \frac{1}{r} \, and \ so \ on.$$

Thus, 1/a, 1/b, 1/c, 1/d,……. form a G.P. with common ratio 1/r.

Theorem 3. If each term of a G.P. be raised to the same power, the resulting terms are in G.P.

Proof. Let a, b, c, d, ... be a G.P. with common ratio r.

Then, b/a=c/b= d/c=……= r

Now, let k be a non-zero number.

Consider, the progression a^k, b^k, c^k, d^k, ...

Clearly, $b^k/a^k = (b/a)^k = r^k$; $c^k/b^k = (c/b)^k = r^k$; $d^k/c^k = (d/c)^k = r^k$; ….

Hence, a^k, b^k, c^k, d^k, …..form a G>P. with common ratio rk.

Ex.1. if a, b, c, d are in G.P., prove that

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2.$$

Sol. Let r be the common ratio of the G.P. a, b, c, d.

Then $b = ar$; $c = ar^2$ and $d = ar^3$.

$\therefore$ L.H.S. $= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6) = a^4r^2(l + r^2 + r^4)^2$;

And, R.H.S. $= (ab + bc + cd)^2 - (a^2 r + a^2r^3 + a^2r^5)^2 = a^4r^2(l + r^2 + r^4)^2$.

Hence, $(a^2 + fe^2 + c^2)(fe^2 + c^2 + d^2) = (ab + bc + cd)^2$.

Ex.2, if a, fe, c, d are in G.P., prove that

$(a^2 + b^2), (b^2 + c^2), (c^2 + d^2)$ are in G.P.

Sol. Let r be the common ratio of the G.P. a**,** b, c, d.

Then, $b = ar$; $c = ar^2$ and $d = ar^3$.

$(a^2 + b^2) = (a^2 + a^2r^2) = a^2(1 + r^2)$; $(b^2 + c^2) = (a^2r^2 + a^2r^4) = a^2r^2(1$
$+ r^2)$;

and $(c^2 + d^2) = (a^2r^4 + a^2r^6) = a^2r^4(l + r^2)$.

Thus, $(b^2 + c^2)^2 = (a^2 + b^2)(c^2 - d^2)$.

This shows that $(a^2 + b^2)$, $(b^2 + c^2)$ and $(c^2 + d^2)$ are in G.P.

Ex.3. If a, b, c, d are in G.P., prove that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P.

Sol. Let r be the common ratio of the G.P. a, b, c, d.

Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

$\therefore (a^n + b^n) = (a^n + a^n r^n) = a^n(1 + r^n)$;

$\{b^n + c^n) = (a^nr^n + a^n r^{2n}) = a^n r^n (1 + r^n)$;

and, $(c^n + d^n) = (a^nr^{2n} + a^n r^{3n}) = a^n r^{2n}(1 + r^n)$.

Thus, $(b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$.

This shows that $(a^n + b^n)$, $(b^n + c^n)$, $(c^n + d^n)$ are in G.P.

3.12 SPECIAL CASES

3.12.1 Sum of first n Natural numbers

Let Sn=1+2+3+4+........................+n.

This is clearly an arithmetic series with a=1, d=1, l=n and the number of terms = n.

∴ Sn = (n/2)/(1=n)= n(n+1)/2

Hence $\sum_{k=1}^{n} n(n + 1)/2$

3.12.2 Sum of the squares of first n Natural numbers

To find $\sum_{k=1}^{n} k2$

Let Sn = 1^{2+} 2^2+ 3^2+.......................+n^2.

Consider the identity $x^3 - (x-1)^3 = 3x^2-3x+1$....................(i)

Putting x=1, 2,3, ..., (n-1), n successively, we get:

$1^3-0^3=3.1^2 - 3.1 +1;$

$2^3-1^3=3.2^{3v}- 3.2 +1;$

$3^3-2^3 = 3.3.^3- 3.3 +1;$

... .

$(n-1)^3 - (n-2)^3 = 3. (n-1)^2 - 3. (n-1) + 1;$

$n^3 - (n-1)^3 = 3.n^2 - 3.n+1$

Adding column wise, we obtain:

$$n^3 = 3 \,.\, [\, 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 \,]$$

$$- 3 \,.\, [\, 1 + 2 + 3 + \ldots + (n-1) + n \,] + [\, 1 + 1 + \ldots \text{ to } n \text{ terms} \,]$$

$$= \left[3 \,.\, \left(\sum_{k=1}^{n} k^2 \right) - 3 \,.\, \left(\sum_{k=1}^{n} k \right) + n \right]$$

$$\therefore\ 3 \,.\, \left(\sum_{k=1}^{n} k^2 \right) = n^3 + 3 \left(\sum_{k=1}^{n} k \right) - n$$

$$= n^3 + \frac{3n\,(n+1)}{2} - n \quad \left[\because\ \sum_{k=1}^{n} k = \frac{n\,(n+1)}{2} \right]$$

$$= \left(\frac{2n^3 + 3n^2 + n}{2} \right) = \frac{n\,(n+1)\,(2n+1)}{2}.$$

$$\therefore\ \left(\sum_{k=1}^{n} k^2 \right) = \frac{n\,(n+1)\,(2n+1)}{6}.$$

3.12.3 Sum of the Cubes of First n Natural Numbers:

To find $\sum_{k=1}^{n} k3$.

Let Sn = $1^3 + 2^3 + 3^3 + 4^3 + \ldots\ldots\ldots\ldots + n^3$.

Consider the identity

$x^4 - (x-1)^4 = 4x3 - 6x2 + 4x - 1$...............................(i)

Putting x= 1, 2, 3, 4,...., (n-1), n successively, we get:

$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1;$

$2^4 - 1^4 = 4.2^3 - 6.2^2 + 4.2 - 1;$

$3^4 - 2^4 = 4.3^3 - 6.3^2 + 4.3 - 1;$

$(n-1)^4 - (n-2)^4 = 4.\,(n-1)^3 - 6.\,(n-1)^2 + 4.\,(n-1) - 1$

$n^4 - (n-1)^4 = 4.\,n^3 - 6.\,n^2 + 4.n - 1$

Adding columnwise, we obtains

$$n^4 = [\,4\,(1^3 + 2^3 + 3^3 + \ldots + n^3) - 6\,(1^2 + 2^2 + 3^2 + \ldots + n^2) + 4\,(1 + 2 + 3 + \ldots + n) - (1 + 1 + \ldots + \text{to } n \text{ terms})\,]$$

$$= \left[\,4\left(\sum_{k=1}^{n} k^3\right) - 6\left(\sum_{k=1}^{n} k^2\right) + 4\left(\sum_{k=1}^{n} k\right) - n\,\right]$$

$$\therefore\; 4\left(\sum_{k=1}^{n} k^3\right) = \left[\,n^4 + 6\left(\sum_{k=1}^{n} k^2\right) - 4\left(\sum_{k=1}^{n} k\right) + n\,\right]$$

$$= \left[\,n^4 + 6\,.\,\frac{n\,(n+1)\,(2n+1)}{6} - 4\,.\,\frac{n\,(n+1)}{2} + n\,\right]$$

$= n[\, n^3 + (n+1)(2n+1) - 2(n+1) + 1]$

$= n(n^3 + 2n^2 + n) = n^2(n+1)$

$\therefore (\sum_{K=1}^{n} K3) = n^2(n+1)2/4 = [n(n+1)/2]^2 = (\sum_{K=1}^{n} k)2$

3.12.4 Sum of Cubes of First n Odd natural Numbers:

Let $Sn = 1^3 + 3^3 + 5^3 + \ldots\ldots\ldots$ to n terms.

Clearly, the nth terms of the series is $(2n-1)^3$.

$$\therefore Sn = \sum_{k=1}^{n} (2k -)3$$

$$\sum_{k=1}^{n} (8k2 - 12k2 + 6k - 1)$$

$$= \left[8\left(\sum_{k=1}^{n} k^3\right) - 12.\left(\sum_{k=1}^{n} k^2\right) + 6\left(\sum_{k=1}^{n} k\right) - n \right]$$

$$[\because\ 1 + 1 + \ldots \text{ to } n \text{ terms} = n\ ]$$

$$= 8 \cdot \frac{1}{4} n^2 (n+1)^2 - 12 \cdot \frac{1}{6} n (n+1)(2n+1) + 6 \cdot \frac{1}{2} n (n+1) - n$$

$$= 2n^2 (n+1)^2 - 2n (n+1)(2n+1) + 3n (n+1) - n$$

$$= n (n+1)(2n^2 - 2n + 1) - n$$

$$= n [(n+1)(2n^2 - 2n + 1) - 1] = n (2n^3 - n)$$

$= n^2 (2n^2 - 1)$.

Ex.1. Find the sum of n terms of the series whose nth term is:

(i) $n(n+3)$ (ii) $(n^2 + 2^n)$

Sol. (i) Sn = $\sum_{k=1}^{n}(k2 + 3k)$ $[\therefore an = (n2 + 3n)]$

$= \sum_{k=1}^{n} k2 + 3 (\sum_{k=1}^{n} k$

$= 1/6\ n (n+1) (2n+1) + 3.\ 1/2n (n+1)$

$= n(n+1)/6 . [(2n+1) + 9]$

$= 1/3n(n+1) (n+5)$.

(ii) $a_n = n^2+2^n$

Putting n= 1, 2, 3, ...n successively, we get

$a^1 = 1^2 + 2^1$;

$a^2 = 2^2 + 2^2$;

$a^3 = 3^2 + 2^3$;

....

$a_n = n^2 + 2^n$

Adding column wise, we get

$$= (t_1 + t_2 + \dots + t_n) = (1^2 + 2^2 + \dots + n^2) + (2 + 2^2 + 2^3 + \dots + 2^n)$$

$$= \left(\sum_{k=1}^{n} k^2 \right) + \left[\frac{2\,(2^n - 1)}{(2-1)} \right]$$

$$= \frac{1}{6} n\,(n+1)\,(2n+1) + 2\,(2^n - 1)\,.$$

Ex.2. *Sum the series :* 3.8 + 6.11 + 9.14 + ... *to n* terms.

Sol. t_n = (nth term of 3, 6, 9, ...) × (nth term of 8, 11, 14, ...)

$$= [\,3 + (n-1) \times 3\,] \times [\,8 + (n-1) \times 3\,] = (3n)\,(3n+5)$$

$$= (9n^2 + 15n)\,.$$

$$\therefore\ S_n = \sum_{k=1}^{n} (9k^2 + 15k)$$

$$= 9 \left(\sum_{k=1}^{n} k^2 \right) + 15 \left(\sum_{k=1}^{n} k \right)$$

$$= 9 \cdot \left[\frac{1}{6} n\,(n+1)\,(2n+1) \right] + 15 \cdot \left[\frac{1}{2} n\,(n+1) \right]$$

$$= \frac{3}{2} n\,(n+1)\,[\,(2n+1) + 5\,]$$

$$= 3n\,(n+1)\,(n+3)\,.$$

Ex.3. *Sum the series* 1.2.3 + 2.3.4 + 3.4.5 + ... *to n terms.*

Sol. t_n = (nth term of 1, 2, 3, ...) × (nth term of 2, 3, 4, ...) × (nth term of 3, 4, 5, ...)

$$= n \times [\,2 + (n-1)\,.\,1\,] \times [\,3 + (n-1)\,.\,1\,]$$

$$= n\,(n+1)\,(n+2) = [\,n^3 + 3n^2 + 2n\,]\,.$$

$$\therefore\ S_n = \sum_{k=1}^{n} (k^3 + 3k^2 + 2k)$$

$$= \left(\sum_{k=1}^{n} k^3 \right) + 3 \left(\sum_{k=1}^{n} k^2 \right) + 2 \left(\sum_{k=1}^{n} k \right)$$

$$= \frac{1}{4} n^2 (n+1)^2 + 3 \cdot \frac{1}{6} n (n+1)(2n+1) + 2 \cdot \frac{1}{2} n (n+1)$$

$$= \frac{1}{4} n (n+1) [\, n (n+1) + 2 (2n+1) + 4 \,]$$

$$= \frac{1}{4} n (n+1)(n+2)(n+3) .$$

Ex.4. *Find the sum of n terms of the series*

$$1^2 + 3^2 + 5^2 + \ldots\ldots \text{ to } n \text{ terms.}$$

Sol. $t_n = [\, 1 + (n-1) \times 2 \,]^2 = (2n-1)^2 = (4n^2 - 4n + 1)$.

$$\therefore \quad S_n = \sum_{k=1}^{n} (4k^2 - 4k + 1) = \left[4 \left(\sum_{k=1}^{n} k^2 \right) - 4 \left(\sum_{k=1}^{n} k \right) + n \right]$$

$$= 4 \cdot \frac{1}{6} n (n+1)(2n+1) - 4 \cdot \frac{1}{2} n (n+1) + n$$

$$= \frac{n}{3} [\, 2 (n+1)(2n+1) - 6 (n+1) + 3 \,]$$

$$= \frac{n}{3} (4n^2 - 1) .$$

Ex.5. *If* S_1, S_2, S_3 *are the sums of first n natural numbers, their squares, their cubes respectively, show that*

$$9 S_2{}^2 = S_3 (1 + 8 S_1) .$$

Sol. We have

$$S_1 = \frac{1}{2} n (n+1) ; \; S_2 = \frac{1}{6} n (n+1)(2n+1)$$

and $\quad S_3 = \frac{1}{4} n^2 (n+1)^2$.

$$\therefore \quad 9 S_2{}^2 = 9 \cdot \left[\frac{n (n+1)(2n+1)}{6} \right]^2 = \frac{1}{4} n^2 (n+1)^2 (2n+1)^2.$$

And, $\quad S_3 (1 + 8 S_1) = \frac{1}{4} n^2 (n+1)^2 \cdot \left[1 + 8 \cdot \frac{1}{2} n (n+1) \right]$

$$= \frac{1}{4} n^2 (n+1)^2 (2n+1)^2.$$

Hence $\quad 9 S_2{}^2 = S_3 (1 + 8 S_1)$.

Ex.6. *Show that*

$$\frac{1 \times 2^2 + 2 \times 3^2 + \ldots + n \times (n+1)^2}{1^2 \times 2 + 2^2 \times 3 + \ldots + n^2 \times (n+1)} = \frac{3n+5}{3n+1} .$$

Sol. nth term of the $N^r = n\,(n+1)^2 = n^3 + 2n^2 + n$;

nth term of the $D^r = n^2\,(n+1) = n^3 + n^2$.

$$\therefore \text{ L.H.S.} = \frac{\sum\limits_{k=1}^{n} (k^3 + 2k^2 + k)}{\sum\limits_{k=1}^{n} (k^3 + k^2)} = \frac{\sum\limits_{k=1}^{n} k^3 + 2\sum\limits_{k=1}^{n} k^2 + \sum\limits_{k=1}^{n} k}{\sum\limits_{k=1}^{n} k^3 + \sum\limits_{k=1}^{n} k^2}$$

$$= \frac{\frac{1}{4} n^2\,(n+1)^2 + 2 \cdot \frac{1}{6} n\,(n+1)\,(2n+1) + \frac{1}{2} n\,(n+1)}{\frac{1}{4} n^2\,(n+1)^2 + \frac{1}{6} n\,(n+1)\,(2n+1)}$$

$$= \frac{n\,(n+1)\,(n+2)\,(3n+5)}{12} \times \frac{12}{n\,(n+1)\,(n+2)\,(3n+1)}$$

[on simplifying]

$$= \frac{3n+5}{3n+1} = \text{R.H.S.}$$

3.13 SUMMARY

- Sequence is a set of numbers written in a particular order. We sometimes write u1 for the first term of the sequence, u2 for the second term, and so on. We write the n-th term as un.

- A series is a sum of the terms in a sequence. If there are n terms in the sequence and we evaluate the sum then we often write Sn for the result, so that

  Sn = u1 + u2 + u3 + . . . + un.

- An arithmetic progression, or AP, is a sequence where each new term after the first is obtained by adding a constant d, called the common difference, to the preceding term. If the first term of the sequence is a then the arithmetic progression isa, a + d, a + 2d, a + 3d, . . .

- The sum of the terms of an arithmetic progression gives an arithmetic series. If the starting value is a and the common difference is d then the sum of the first n terms is

  $$S_n = n/2(2a + (n - 1)d).$$

- If we know the value of the last term ℓ instead of the common difference d then we can write the sum as

  $$S_n = n/2\,(a + \ell).$$

- A geometric progression, or GP, is a sequence where each new term after the first is obtained by multiplying the preceding term by a constant r, called the common ratio. If the first term of the sequence is a then the geometric progression is a, ar, ar2, ar3,....

- The sum of the terms of a geometric progression gives a geometric series. If the starting value is a and the common ratio is r then the sum of the first n terms is

  $$S_n = \frac{a(1 - r^n)}{1 - r}$$

- The sum to infinity of a geometric progression with starting value a and common ratio r is given by:

  $$\mathbf{S}\propto = a/1\text{-}r$$

- The sum of the terms in a geometric progression has a limit (note that this is summing together an infinite number of terms). A series like this has a limit partly because each successive term we are adding is smaller and smaller (but this fact in itself is not enough to say that the limiting sum exists). When the sum of a geometric series has a limit we say that S1 exists and we can find the limit of the sum.

3.14 SELF ASSESSMENT EXERCISES

1. A sequence is given by the formula un = 3n + 5, for n = 1, 2, 3, ... Write down the first five terms of this sequence.

2. Write down S1, S2, . . . , Sn for the sequences.
   (a) 1, 3, 5, 7, 9, 11;
   (b) 4, 2, 0, −2, −4.

3. Write down the first five terms of the AP with first term 8 and common difference 7.

4. Write down the first five terms of the AP with first term 2 and common difference −5.

5. What is the common difference of the AP 11,−1,−13,−25, . . . ?

6. Find the 17th term of the arithmetic progression with first term 5 and common difference

7. Find the sum of the series.....1 + 3 · 5 + 6 + 8 · 5 + . . . + 101 .

8. An arithmetic progression has 3 as its first term. Also, the sum of the first 8 terms is twice the sum of the first 5 terms. Find the common difference.

9. Find the sum of the first 23 terms of the AP 4,−3,−10, ...

10. An arithmetic series has first term 4 and common difference 1

11. Find:

(i) the sum of the first 20 terms,

(ii) the sum of the first 100 terms.

(iii) Find the sum of the arithmetic series with first term 1, common difference 3, and last term 100.

(iv) The sum of the first 20 terms of an arithmetic series is identical to the sum of the first terms. If the common difference is −2, find the first term.

12. Write down the first five terms of the geometric progression which has first term 1 and common ratio.

13. Find the 10th and 20th terms of the GP with first term 3 and common ratio 2.

14. Find the 7th term of the GP 2,−6, 18.....

15. Find the sum of the geometric series 2 + 6 + 18 + 54 + . . . where there are 6 terms in the series.

16. Find the sum of the geometric series 8 − 4 + 2 − 1 + . . . where there are 5 terms in the series.

17. How many terms are there in the geometric progression 2, 4, 8, . . . 128 ?

18. How many terms in the geometric progression 1, 1 · 1, 1 · 21, 1 · 331, . . will be needed so that the sum of the first n terms is greater than 20?

19. Find the sum of the first five terms of the GP with first term 3 and common ratio 2.

20. Find the sum of the first 20 terms of the GP with first term 3 and common ratio 1.5.

21. How many terms in the GP 4, 3.6, 3.24, . . . are needed so that the sum exceeds 35?

22. If (p + l)th term of an A.P. is twice the (q + l)th term, prove that the (3 p + l)th term is twice the (p + q + l)th term.

23. Find the 15th term from the end of the A.P. 3, 5, 7, 9, ..., 201.

24. How many numbers of two digits are divisible by 7 ?

25. The first and the last terms of an A.P. are a and l respectively. Show that the sum of nth term from the beginning and the nth term from the end is (a + l).

26. Three numbers are in A.P. If the sum of these numbers be 27 and the product 648, find the numbers.

27. The sum of three consecutive terms of an A.P. is 21. If the sum of the squares of these terms be 165, find these terms.

28. The angles of a quadrilateral are in A.P. whose common difference is 10°. Find the angles.

29. The sum of first 13 terms of an A.P. is 21 and the sum of first 21 terms is 13. Show that the sum of first 34 terms is - 34.

30. Find the sum of first n natural numbers.

31. Find the sum of all natural numbers between 1 and 100, which are divisible by 3.

32. Find the sum of all odd numbers between 100 and 200.

33. Find the sum of all integers between 84 and 719, which are multiples of 5.

34. Find the sum of all integers between 50 and 500 which are divisible by 7.

35. The sum of three numbers in G.P. is 21 and the sum of their squares is 189. Find the numbers.

36. The product of three numbers in G.P. is 216 and the sum of the products of the numbers, taken in pairs is 156. Find the numbers.

37. Three numbers are in A.P. and their sum is 15. If 1, 3, 9 be added to them respectively, they form a G.P. Find the numbers.

38. The product of first three terms of a G.P. is 1000. If 6 is added to its second term and 7 is added to its third term, the terms become in A.P. Find the G.P.

39. The sum of four numbers in G.P. is 60 and the arithmetic mean of the first and the last is 18. Find the numbers.

40. The sum of three numbers ***a, b, c*** in A.P. is 18. If a and b are each increased by 4 and c is increased by 36, the new numbers form a G.P. Find a, b, c.

3.15 SUGGESTED READING

- Mathematics, R.S. Aggarwal, Bharati Bhawan, Patna.
- Business Mathematics, Kavita Choudhary.
- Mathematics, R.D. Sharma.

STRUCTURE

4.1 INTRODUCTION

Sir ARTHUR CAYLEY (1821-1895) of England was the first Mathematician to introduce the term Matrix in the year 1858. But in the present day applied Mathematics in overwhelmingly large majority of cases it is used, as a notation to represent a large number of simultaneous equations in a compact and convenient manner. Matrix Theory has its applications in Operations Research, Economics and Psychology. Apart from the above, matrices are now indispensible in all branches of Engineering, Physical and Social Sciences, Business Management, Statistics and Modern Control systems.

In business mathematics, one comes across mathematical objects, other than numbers, which combined with one another through one or more compositions, whose laws are more or less identical with those of the compositions with number. Matrices which constitute the subject of study in this and two next lessons are mathematical objects of this type.

Matrix has had its origin in various types of linear problems, the most important of which concerns the nature of solutions of any given system of linear equations and linear transformations.

4.2 OBJECTIVE

After reading this lesson you would be able:

- To understand the concept of matrices and determinants.
- To know about the various properties of determinants.
- To make you aware about different types of matrices
- To learn multiplication of matrix by numbers.
- To apply the learned concepts in practical situation.

4.3 CONCEPT OF MATRIX

4.3.1 Definition

A system of $m \ n$ numbers arranged in the form of an ordered set of m rows, each row consisting of an ordered set of n numbers, is called an m **x** n (to be read as m by $r_i)$ matrix.

If there are m rows and n columns in the array, the matrix is said to be of order (m, n) or an m **X** n (to be read as m by n) matrix. We say that the matrix is of the type m **x** n. If $m = n$, then the matrix is said to be of order n.

There are four different symbols [ ], ("), || ||, { } for enclosing the numbers of the array. We shall apply mostly the first two symbols.

For e.g.

$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$; $\begin{bmatrix} 6 & -3 & 4 \\ 7 & 5 & 2 \end{bmatrix}$; $\begin{bmatrix} 6 & 5 & 5 \\ 1 & 3 & -4 \\ 4 & 2 & 6 \end{bmatrix}$ are matrices of order 2*2, 2*3, 3*3 respectively.

For convenience, we denote matrices by capital letters A, B, C The element in the ith row and jth column of A is denoted by a_{ij}.

we write A = [a_{ij}] where i varies from 1 to m and j varies from 1 to. n.

In other words, a rectangular array of numbers or functions represented by the symbol,

$$\begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & 23 \\ a31 & a32 & a33 \end{pmatrix}$$ is called a matrix.

The numbers or functions aij of this array are called elements, may be real or complex numbers, where as m and n are positive integers, which denotes the number of Rows and number of Columns.

4.3.2 Equality of matrices

Two matrices A and B are said to be equal if and only if:

i) A and B are of the same type.

ii) All the elements of A are the same as the corresponding elements of B.

OR

Two matrices A = [a_{ij}], B = [b_{ij}] of the same type are said to be equal if, aij = bij.

4.3.3 Special Types of Matrices

A. Row Matrix: An m $\times$ n matrix is called a row matrix i£ m = 1

For example $[1 \quad 0 \quad 2 + 3i]$ is a row matrix.

B. Column Matrix: An m $\times$ n matrix is called a column matrix if $n = 1$.

For example $\begin{matrix} 3 \\ 5 \\ 7 \end{matrix}$ is a column matrix.

C. Square Matrix: An m $\times$ n matrix is called a square matrix if $m = n$.

For example $\begin{bmatrix} 2 & 0 \\ 9 & -3 \end{bmatrix}$ is a square matrix.

The above square matrix is of order 2 and is sometimes called 2-rowed matrix.

D. Rectangular Matrix: An matrix which is not a square matrix, is called a rectangular matrix.

E. Zero Matrix: A matrix each of whose elements is zero is called a zero matrix and is denoted by O. Zero matrix is also called a Null matrix.

Note on diagonal elements of a matrix

The elements a_{ij} of any square matrix $A = [a_{ij}]$ for which $i = j$ are called the diagonal elements of the matrix. Diagonal elements are said to constitute the main diagonal or principal diagonal or simply a diagonal. The elements which lie on a line perpendicular to the diagonal are said to constitute the secondary diagonal.

For example, in the matrix $\begin{bmatrix} 2 & 7 \\ 0 & -9 \end{bmatrix}$ Main diagonal consists of 2 and -9 while secondary diagonal consists of 7 and 0. The sum of the diagonal elements of square matrix is called the trace of A.

F. Diagonal Matrix: A square matrix with all its non-diagonal elements as zero is called a diagonal matrix. Therefore, if A = [aij] is a diagonal matrix, then aij = 0 whenever i$\neq j$. For example, $\begin{bmatrix} x1 & 0 & 0 \\ 0 & x2 & 0 \\ 0 & 0 & x3 \end{bmatrix}$ is a diagonal matrix.

For convenience it is written as diag. [x1, x2, x3].

A diagonal matrix with diagonal elements d1, d2,……dn will bw generally denoted by,,

Diag. [d1, d2, ……dn].

G. Scalar Matrix: A diagonal matrix all of whose diagonal elements are equal is called a scalar matrix.

H. Unit matrix: A scalar matrix all of .whose diagonal elements are equal to unity is called a Unit matrix and is denoted by I,, , if it is of order n. Unit matrix is also called an Identity Matrix.

I. Triangular_Matrix: If every element above or below the diagonal is zero, the matrix is said to be a triangular matrix.

OR

A matrix A = [aij] is called a triangular matrix if aij = 0, whenever $i > j$ or $i < j$.

If $cij = 0$ when $i > j$, the matrix is called upper triangular and if $a_u = 0$ when (i$< j$) the matrix is called a lower triangular one.

For example;

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$ **is a triangular matrix,**

$$\begin{bmatrix} 4 & 6 & 7 \\ 0 & 5 & 9 \\ 0 & 0 & 2 \end{bmatrix}$$ **is an upper triangular matrix**

and $$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 5 & 0 \\ 8 & 6 & 3 \end{bmatrix}$$ **is a lower triangular matrix.**

4.4 ALGEBRA OF MATRICES

4.4.1 Addition of Matrices

If A = [a i j] , B = [b i j] be two matrices of the same type $m \times n$, then their sum A + B is defined as the matrix A + B = [aij + bij]
We say that C = [c i j] is the sum of A and B if c i j = a i j + b i j .
From the above definition it is clear that two matrices of the same type are added by adding their corresponding elements.
Two matrices which can be added are said to be conformable for addition. For example, if

$$A = \begin{bmatrix} 2 & 0 & -9 \\ 0 & 7 & 3 \end{bmatrix};\ B = \begin{bmatrix} 1 & 9 & -3 \\ 0 & -5 & -7 \end{bmatrix}$$

$$\text{Then } A + B = \begin{bmatrix} 2+1 & 0+9 & -9-3 \\ 0+0 & 7-5 & 3-7 \end{bmatrix} = \begin{bmatrix} 2 & 9 & -12 \\ 0 & 2 & -4 \end{bmatrix}$$

A + B is not defined when A and B are not of the same type.

4.4.2 Subtraction of Matrices

If A = [a, j] , B = [bj j] are two matrices of the same type, then difference is defined as the matrix C = [c,-7-] where c, ;- = $aij - bij$
For example,

$$\text{If } A = \begin{bmatrix} 1 & 7 & -3 \\ 2 & 0 & 5 \\ 3 & 0 & 7 \end{bmatrix},\ B = \begin{bmatrix} -1 & -7 & 5 \\ 9 & -3 & 2 \\ 5 & 7 & -4 \end{bmatrix}$$

$$\text{then } A - B = \begin{bmatrix} 1-(-1) & 7-(-7) & -3-5 \\ 2-9 & 0-(-3) & 5-2 \\ 3-5 & 0-7 & 7-(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 14 & -8 \\ -7 & 3 & 3 \\ -2 & -7 & 11 \end{bmatrix}$$

4.4.3 Scalar Multiplication

If each element of a matrix A is multiplied by a Scalar k (we call k a scalar to distinguish it form [k] which is 1 × 1 matrix), then the resulting matrix kA = Ak is called the scalar multiple of A by k.

4.4.4 Negative of a Matrix

Negative of a matrix: Negative of a matrix A $i.e.$ A is the matrix obtained from A by multiplying each of its elements by - 1.

Proof 1: If A, B and C are three matrices of the same order, then:

(i) A + B = B + A. (addition is commutative)

(ii) A + (B + C) = (A + B) + C (addition is associative).

Verification

$$(i) \quad \text{Let } A = \begin{bmatrix} 2 & 5 \\ 6 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & -7 \\ 4 & 5 \end{bmatrix}$$

$$\therefore \quad A + B = \begin{bmatrix} 2 & 5 \\ 6 & -9 \end{bmatrix} + \begin{bmatrix} 8 & -7 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2+8 & 5-7 \\ 6+4 & -9+5 \end{bmatrix}$$

$$\therefore \quad A + B = \begin{bmatrix} 10 & -2 \\ 10 & -4 \end{bmatrix} \qquad \ldots(1)$$

$$B + A = \begin{bmatrix} 8 & -7 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ 6 & -9 \end{bmatrix} = \begin{bmatrix} 8+2 & -7+5 \\ 4+6 & 5-9 \end{bmatrix}$$

$$\therefore \quad B + A = \begin{bmatrix} 10 & -2 \\ 10 & -4 \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2), $A + B = B + A$

(ii)Verification

(ii) Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 5 \\ -6 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

$$B + C = \begin{bmatrix} -2 & 5 \\ -6 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -2+2 & 5+4 \\ -6+6 & 3+8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 9 \\ 0 & 11 \end{bmatrix}$$

$$\therefore \quad A + (B + C) = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 9 \\ 0 & 11 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+9 \\ 4+0 & 5+11 \end{bmatrix}$$

$$\therefore \quad A + (B + C) = \begin{bmatrix} 1 & 12 \\ 4 & 16 \end{bmatrix} \qquad \ldots(1)$$

$$A + B = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} -2 & 5 \\ -6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2 & 3+5 \\ 4-6 & 5+3 \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ -2 & 8 \end{bmatrix}$$

$$\therefore \quad (A + B) + C = \begin{bmatrix} -1 & 8 \\ -2 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} -1+2 & 8+4 \\ -2+6 & 8+8 \end{bmatrix}$$

$$\therefore \quad (A + B) + C = \begin{bmatrix} 1 & 12 \\ 4 & 16 \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2), we get,

$$A + (B + C) = (A + B) + C$$

If A and B are two matrices of the same order and K and l are numbers, then

(i) K(A+B) = KA + KB
(ii) (K+1) A = KA + LA

Verification:

(i) Let k= 5, $A = \begin{bmatrix} 2 & 5 \\ 7 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 4 & -3 \\ -2 & 6 \end{bmatrix}$

$$A + B = \begin{bmatrix} 2 & 5 \\ 7 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -3 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 7 \end{bmatrix}$$

$$\therefore \mathbf{A} + \mathbf{B} = \begin{bmatrix} 6 & 2 \\ 5 & 7 \end{bmatrix}$$

$$\therefore \mathbf{k(A+B)} = \mathbf{5}\begin{bmatrix}\mathbf{6} & \mathbf{2}\\ \mathbf{5} & \mathbf{7}\end{bmatrix}$$

⇨ K(A+B) = $\begin{bmatrix}\mathbf{30} & \mathbf{10}\\ \mathbf{25} & \mathbf{35}\end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(\mathbf{1})$

K(A) = 5$\begin{bmatrix}\mathbf{2} & \mathbf{5}\\ \mathbf{7} & \mathbf{1}\end{bmatrix} = \begin{bmatrix}\mathbf{10} & \mathbf{25}\\ \mathbf{25} & \mathbf{5}\end{bmatrix}$

K(B) = 5$\begin{bmatrix}\mathbf{4} & \mathbf{-3}\\ \mathbf{-2} & \mathbf{6}\end{bmatrix} = \begin{bmatrix}\mathbf{20} & \mathbf{-15}\\ \mathbf{-10} & \mathbf{30}\end{bmatrix}$

$$\therefore kA + KB = \begin{bmatrix}10 & 25\\ 25 & 5\end{bmatrix} + \begin{bmatrix}20 & -15\\ -10 & 30\end{bmatrix} = \begin{bmatrix}30 & 10\\ 25 & 35\end{bmatrix}$$

⇨ kA + kB = $\begin{bmatrix}\mathbf{30} & \mathbf{10}\\ \mathbf{25} & \mathbf{35}\end{bmatrix}$

From (1) and (2), we get,

K (A+B) = kA + kB

(ii) let k= 2, l= 4, A = $\begin{bmatrix}\mathbf{3} & \mathbf{2} & \mathbf{7}\\ \mathbf{1} & \mathbf{-5} & \mathbf{4}\\ \mathbf{-2} & \mathbf{4} & \mathbf{-8}\end{bmatrix}$

(k+l)A = (2+4) $\begin{bmatrix}3 & 2 & 7\\ 1 & -5 & 4\\ -2 & 4 & -8\end{bmatrix} = 6\begin{bmatrix}3 & 2 & 7\\ 1 & -5 & 4\\ -2 & 4 & -8\end{bmatrix}$

$$\therefore (k+l)A = \begin{bmatrix}18 & 12 & 42\\ 6 & -30 & 24\\ -12 & 24 & -48\end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

$$kA = 2\begin{bmatrix}3 & 2 & 7\\ 1 & -5 & 4\\ -2 & 4 & -8\end{bmatrix} = \begin{bmatrix}6 & 4 & 14\\ 2 & -10 & 8\\ -4 & 8 & -16\end{bmatrix}$$

$$lA = 4\begin{bmatrix}3 & 2 & 7\\ 1 & -5 & 4\\ -2 & 4 & -8\end{bmatrix} = \begin{bmatrix}12 & 8 & 28\\ 4 & -20 & 16\\ -8 & 16 & -32\end{bmatrix}$$

$$\therefore kA + lA = \begin{bmatrix} 6 & 4 & 14 \\ 2 & -10 & 8 \\ -4 & 8 & -16 \end{bmatrix} + \begin{bmatrix} 12 & 8 & 28 \\ 4 & -20 & 16 \\ -8 & 16 & -32 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 12 & 42 \\ 6 & -30 & 24 \\ -12 & 24 & -48 \end{bmatrix}$$

$$\therefore kA + lA = \begin{bmatrix} 18 & 12 & 42 \\ 6 & -30 & 24 \\ -12 & 24 & -48 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2), we get,

(k + l)A = kA + l A

Example 1: Construct a 2x4 matrix A = [aij] whose elements are aij = 2i – j

Sol. Here aij = 2 i - j

$\therefore$ $a_{11} = 2(1) - 1 = 2 - 1 = 1$

$a_{12} = 2(l) - 2 = 2 - 2 = 0$

$a_{13} = 2(l) - 3 = 2 - 3 = -l$

$a_{14} = 2(l) - 4 = 2 - 4 = -2$

$a_{21} = 2(2) - 1 = 4 - 1 = 3$

$a_{21} = 2(2) - 2 = 4 - 2 = 2$

$a_{23} = 2(2) - 3 = 4 - 3 = 1$

$a_{24} = 2(2) - 4 = 4 - 4 = 0$

$\therefore$ A = $\begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix}$.

Example 2: if $\begin{bmatrix} \boldsymbol{x-y} & \boldsymbol{2x+z} \\ \boldsymbol{2x-y} & \boldsymbol{3z+w} \end{bmatrix} = \begin{bmatrix} \boldsymbol{-1} & \boldsymbol{5} \\ \boldsymbol{0} & \boldsymbol{13} \end{bmatrix}$, $\boldsymbol{find\ x, y, z, w}$.

Sol: we are given that

$$\begin{bmatrix} \boldsymbol{x-y} & \boldsymbol{2x+z} \\ \boldsymbol{2x-y} & \boldsymbol{3z+w} \end{bmatrix} = \begin{bmatrix} \boldsymbol{-1} & \boldsymbol{5} \\ \boldsymbol{0} & \boldsymbol{13} \end{bmatrix}$$

From the definition of equality of matrices, we_have.

$x - y = -1 \ldots\ldots.(1)$

$2x - y = 0 \ldots\ldots(2)$

$2x + z = 5 \ldots\ldots(3)$

$3z + w = 13 \ldots..(4)$

Subtracting (1) from (2), $x = 1$

$\therefore$ From (l), $1 - y = -1 \Rightarrow y = 2$

From (3), $2 + z = 5 \Rightarrow z = 3$

From (4), $9 + w = 13 \Rightarrow w = 4$.

$$\therefore x = 1, y = 2, z = 3, w = 4.$$

Example 3: If A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}, find\ 2A - B.$

Sol: A=$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$

$$\therefore 2A = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix}$$

$$\therefore 2A - B = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$$

Example 4: find X and Y if X + Y = $\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}, X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Sol: we have X + Y = $\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \ldots\ldots\ldots..(1)$

And X - Y = $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$

Adding (1) and (2), we get,

$2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}, \therefore X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$

$$subtracting\ 2\ form\ 1, we\ get,$$

$$2Y = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$$

$$\therefore Y = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

4.4.5 Product of Matrices

Let A = [aij] , B = [bjk] be two matrices of m X n and *n X p* types respectively *i.e.* the number of columns of A is the same as the number of rows of B. The ranges of three suffixes *i, j, k* are 1 to *m,* 1 to *n,* 1 to *p* respectively.

Then the matrix C = [cik] where $c_{ik} = \quad b_{jk}$ is called the product matrix AB of the matrices A and B taken in this order. According to definition of product of matrices, the (i,k)th element of AB is the sum of the products of the elements in the *i* th row of A with the corresponding elements in the k th column of B.

In the product AB, A is called pre factor and B is called post factor. It should be kept in mind that multiplication is only possible if the number of rows of post factor is the same as the number of columns of pre factor. If this condition is true, we say that A and B are conformable for multiplication.

Note : While dealing with product of matrices, student should keep in mind that:

(i) Product AB may be defined where as BA may not be defined. For example if A is 4 X 5 and B is 5 X 6 matrix then AB is defined as the number of columns of A is the same as the number of rows of B, but the product BA is not defined as number of columns of B is not the same as the number of rows of A.

(ii) If AB and B A are both defined, even then AB may not be equal to B A. The above two facts will be illustrated in examples.

Rule to get the product C of two matrices A and B.

Let C = AB

To obtain the matrix C, we proceed as follows:

We multiply the elements of the first row of A by the corresponding elements of the first column of B and add. This is the first element in the first row of C. Then we multiply the elements of the first -row of A by the corresponding element of the second column of B and add. This is the second element in the first row of C. Similarly we obtain the remaining elements of the first row of C.

Proceeding as above, we obtain the elements of the remaining rows of C.

4.4.6 Associative Law For Multiplication

If A, B, C be three matrices of order $m \times n$, $n \times p$, $p \times q$ respectively, then (AB) C = A (BC).

Verification

$$\text{Let } A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \; B = \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \; C = \begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} (2)(4)+(1)(7) \\ (3)(4)+(2)(7) \end{bmatrix} = \begin{bmatrix} 15 \\ 26 \end{bmatrix}$$

$$(AB)\,C = \begin{bmatrix} 15 \\ 26 \end{bmatrix}\begin{bmatrix} 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 30 & 60 & 90 \\ 52 & 104 & 156 \end{bmatrix} \qquad \ldots(1)$$

$$BC = \begin{bmatrix} 4 \\ 7 \end{bmatrix}\begin{bmatrix} 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 8 & 16 & 24 \\ 14 & 28 & 42 \end{bmatrix}$$

$$A\,(BC) = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 8 & 16 & 24 \\ 14 & 28 & 42 \end{bmatrix} = \begin{bmatrix} 16+14 & 32+28 & 48+42 \\ 24+28 & 48+56 & 72+84 \end{bmatrix}$$

$$\therefore \quad A\,(BC) = \begin{bmatrix} 30 & 60 & 90 \\ 52 & 104 & 156 \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2), we get ,

$(AB)\,C = A\,(BC)$.

4.4.7 Left Distributive Law

If A, B, C be three matrices of order $m \times n$, $n \times p$, $p \times q$ respectively, then $A\,(B + C) = AB + AC$.

Verification

$$\text{Let } A = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}, C = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 7 & 10 & 13 \end{bmatrix}$$

$$A(B + C) = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 3 & 6 & 9 \\ 7 & 10 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 6+35 & 12+50 & 18+65 \\ 18+21 & 36+30 & 54+39 \end{bmatrix}$$

$$\therefore \quad A(B + C) = \begin{bmatrix} 41 & 62 & 83 \\ 39 & 66 & 93 \end{bmatrix} \qquad \ldots(1)$$

$$AB = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix} = \begin{bmatrix} 4+25 & 6+30 & 8+35 \\ 12+15 & 18+18 & 24+21 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 36 & 43 \\ 27 & 36 & 45 \end{bmatrix}$$

$$AC = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 2+10 & 6+20 & 10+30 \\ 16+6 & 18+12 & 30+18 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 26 & 40 \\ 12 & 30 & 48 \end{bmatrix}$$

$$\therefore \quad AB + AC = \begin{bmatrix} 29 & 36 & 43 \\ 27 & 36 & 45 \end{bmatrix} + \begin{bmatrix} 12 & 26 & 40 \\ 12 & 30 & 48 \end{bmatrix}$$

$$= \begin{bmatrix} 29+12 & 36+26 & 43+40 \\ 27+12 & 36+30 & 45+48 \end{bmatrix}$$

$$\therefore \quad AB + AC = \begin{bmatrix} 41 & 62 & 83 \\ 39 & 66 & 93 \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2), we get,

$$A(B + C) = AB + AC.$$

4.3.1 Right Distributive Law

Prove that (B + C) A = BA + CA

Wherever both sides of the equality are defined.

Verification

$$\text{Let } A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix}, C = \begin{bmatrix} -1 & 5 \\ 2 & -3 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1-1 & -2+5 \\ -5+2 & 2-3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & -1 \end{bmatrix}$$

$$(B + C)\, A = \begin{bmatrix} 0 & 3 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0+9 & 0+3 \\ -6-3 & -12-1 \end{bmatrix}$$

$$\therefore\ (B + C)\, A = \begin{bmatrix} 9 & 3 \\ -9 & -13 \end{bmatrix} \qquad \ldots(1)$$

$$BA = \begin{bmatrix} 1 & -2 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2-6 & 4-2 \\ -10+6 & -20+2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -4 & -18 \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -2+15 & -4+5 \\ 4-9 & 8-3 \end{bmatrix} = \begin{bmatrix} 13 & 1 \\ -5 & 5 \end{bmatrix}$$

$$BA + CA = \begin{bmatrix} -4 & 2 \\ -4 & -18 \end{bmatrix} + \begin{bmatrix} 13 & 1 \\ -5 & 5 \end{bmatrix} = \begin{bmatrix} -4+13 & 2+1 \\ -4-5 & -18+5 \end{bmatrix}$$

$$\therefore\ BA + CA = \begin{bmatrix} 9 & 3 \\ -9 & -13 \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2), we get,

$$(B + C)\, A = BA + CA.$$

4.4.9 Positive Integral Powers of Matrices

If A is any square matrix, we write A^2 for AA. Now by Associative Law,

$A^2A = (AA)\,A = A\,(AA) = AA^2$ and we write A^2A and AA^2 as A^3.
In general A. A. A... to *m* factors is denoted by A^m ,
Now we have, $A^m \,.\, A^n = (A.A...\text{to } m \text{ factors})\,(AA...\text{to } n \text{ factors})$

$= A.A..$ to $(m + n)$ factors
$A^m . A^n = A^{m+n}$

Similarly $\quad (A^m)^n = A^{mn}$.

Example 1: Compute AB if:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 7 & 5 \end{bmatrix}$$

Sol: We have

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 7 & 5 \end{bmatrix}$$

Since A has 3 columns and B has 3 rows.

Therefore AB is defined.

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 5 & 3 \\ 3 & 6 & 4 \\ 4 & 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+6+12 & 5+12+21 & 3+8+15 \\ 8+15+24 & 20+30+42 & 12+20+30 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 38 & 26 \\ 47 & 92 & 62 \end{bmatrix}$$

Example2:If

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & 0 \\ 5 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 & -1 \\ 0 & 5 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

Then find AB and BA. Is AB= BA? What conclusion do you draw?

Sol. $A = \begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & 0 \\ 5 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 & -1 \\ 0 & 5 & 2 \\ 1 & -2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & 0 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} -3 & 2 & -1 \\ 0 & 5 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (2)(-3)+(-1)(0)+(3)(1) & (2)(2)+(-1)(5)+(3)(-2) & (2)(-1)+(-1)(2)+(3)(1) \\ (-3)(-3)+(2)(0)+(0)(1) & (-3)(2)+(2)(5)+(0)(-2) & (-3)(-1)+(2)(2)+(0)(1) \\ (5)(-3)+(1)(0)+(-1)(1) & (5)(2)+(1)(5)+(-1)(-2) & (5)(-1)+(1)(2)+(-1)(1) \end{bmatrix}$$

$$= \begin{bmatrix} -6+0+3 & 4-5-6 & -2-2+3 \\ 9+0+0 & -6+10+0 & 3+4+0 \\ -15+0-1 & 10+5+2 & -5+2-1 \end{bmatrix} = \begin{bmatrix} -3 & -7 & -1 \\ 9 & 4 & 7 \\ -16 & 17 & -4 \end{bmatrix}$$

Also $BA = \begin{bmatrix} -3 & 2 & -1 \\ 0 & 5 & 2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ -3 & 2 & 0 \\ 5 & 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -6-6-5 & 3+4-1 & -9+0+1 \\ 0-15+10 & 0+10+2 & 0+0-2 \\ 2+6+5 & -1-4+1 & 3-0-1 \end{bmatrix}$

$= \begin{bmatrix} -17 & 6 & -8 \\ -5 & 12 & -2 \\ 13 & -4 & 2 \end{bmatrix}$

$\therefore\ AB \neq BA$

We conclude that multiplication of matrices A and B is not commutative.

Example 3. If $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, find A^2. Hence find the matrix A^6.

Sol. Here $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\therefore\quad A^2 = AA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0+0+1 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 1+0+0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Now $A^6 = (A^2)^3 = (I)^3 = I$

$\therefore\quad A^6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 4: Write as single matrix

$$[1 \quad -2 \quad 3]\begin{bmatrix} 2 & -1 & 5 \\ 0 & 2 & 4 \\ -7 & 5 & 0 \end{bmatrix} - [2 \quad -5 \quad 7]$$

Sol. $$[1 \quad -2 \quad 3]\begin{bmatrix} 2 & -1 & 5 \\ 0 & 2 & 4 \\ -7 & 5 & 0 \end{bmatrix} - [2 \quad -5 \quad 7]$$

$$= [2-0-21 \quad -1-4+15 \quad 5-8+0] - [2 \quad -5 \quad 7]$$

$$= [-19 \quad 10 \quad -3] - [2 \quad -5 \quad 7]$$

$$= [-19-2 \quad 10+5 \quad -3-7] = [-21 \quad 15 \quad -10]$$

Example 5: Find the Value of x Such That

$$[1 \quad 1 \quad x]\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = O.$$

Sol. $$[1 \quad 1 \quad x]\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = O$$

$$\Rightarrow \quad [1 \quad 1 \quad x]\begin{bmatrix} 1+0+2 \\ 0+2+1 \\ 2+1+0 \end{bmatrix} = O \qquad \Rightarrow \quad [1 \quad 1 \quad x]\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = O$$

$$\Rightarrow \quad [3+3+3x] = [0] \qquad \Rightarrow \quad [3x+6] = [0]$$

$$\Rightarrow \quad 3x+6=0 \qquad \Rightarrow \quad 3x=-6$$

$$\Rightarrow \quad x=-2.$$

Example 6: Let f(x) = x2 – 5x + 6. Find f(A) if

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}.$$

$$\therefore \quad A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1-0 & 1-3+0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\therefore \quad A^2 - 5A + 6I$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & 5 & -15 \\ -5 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10+6 & -1+0+0 & 2-5+0 \\ 9-10+0 & -2-5+6 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2+0+6 \end{bmatrix}$$

$$\therefore \quad f(A) = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix} \qquad [\because \text{ of } (1)]$$

Example 7. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, then show that

$$A^3 - 6A^2 - 3A + 18I = O.$$

Sol. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 1+4+9 & 2+6+3 & 3+2+6 \\ 2+6+3 & 4+9+1 & 6+3+2 \\ 3+2+6 & 6+3+2 & 9+1+4 \end{bmatrix} = \begin{bmatrix} 14 & 11 & 11 \\ 11 & 14 & 11 \\ 11 & 11 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 11 & 11 \\ 11 & 14 & 11 \\ 11 & 11 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 14+22+33 & 28+33+11 & 42+11+22 \\ 11+28+33 & 22+42+11 & 33+14+22 \\ 11+22+42 & 22+33+14 & 33+11+28 \end{bmatrix} = \begin{bmatrix} 69 & 72 & 75 \\ 72 & 75 & 69 \\ 75 & 69 & 72 \end{bmatrix}$$

Consider $A^3 - 6A^2 - 3A + 18I$

$$=\begin{bmatrix} 69 & 72 & 75 \\ 72 & 75 & 69 \\ 75 & 69 & 72 \end{bmatrix} - 6\begin{bmatrix} 14 & 11 & 11 \\ 11 & 14 & 11 \\ 11 & 11 & 14 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 69 & 72 & 75 \\ 72 & 75 & 69 \\ 75 & 69 & 72 \end{bmatrix} - \begin{bmatrix} 84 & 66 & 66 \\ 66 & 84 & 66 \\ 66 & 66 & 84 \end{bmatrix} - \begin{bmatrix} 3 & 6 & 9 \\ 6 & 9 & 3 \\ 9 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$=\begin{bmatrix} 69-84-3+18 & 72-66-6+0 & 75-66-9+0 \\ 72-66-6+0 & 75-84-9+18 & 69-66-3+0 \\ 75-66-9+0 & 69-66-3+0 & 72-84-6+18 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore\ A^3 - 6A^2 - 3A + 18I = O.$

Example 8: Give example to show that

(i) AB = O even if A $\neq$ O, B $\neq$O.

(ii) AB = O but BA $\neq \boldsymbol{O}$

(iii) AB $\neq \boldsymbol{O}$ ***but BA*** $= \boldsymbol{O}$

(iv) AB = O BA where A $\neq \boldsymbol{O}, \boldsymbol{B} \neq \boldsymbol{O}$

(v) AB = AC but B $\neq \boldsymbol{C}$

(vi) (A+B)2 = A2 +B2 + 2AB

Sol. (*i*) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

$\therefore \quad AB = O$ even if $A \neq O$, $B \neq O$.

(*ii*) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$BA = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 3+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = O$$

$\therefore \quad AB = O$, but $BA \neq O$.

(*iii*) Let $A = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 3+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \neq O$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\therefore \quad AB = O$, but $BA \neq O$.

(*iv*) Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$BA = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\therefore \quad AB = O = BA$ where $A \neq O$, $B \neq O$.

(*v*) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\therefore \quad AB = AC$, but $B \neq C$.

(*vi*) Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

$\therefore \quad A + B = \begin{bmatrix} 1+3 & 2-4 \\ -2+4 & 1+3 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix}$

$\therefore \quad (A+B)^2 = \begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 16-4 & -8-8 \\ 8+8 & -4+16 \end{bmatrix} = \begin{bmatrix} 12 & -16 \\ 16 & 12 \end{bmatrix}$

$\therefore \quad (A+B)^2 = \begin{bmatrix} 12 & -16 \\ 16 & 12 \end{bmatrix}$ ...(1)

$A^2 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1-4 & 2+2 \\ -2-2 & -4+1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$

$B^2 = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 9-16 & -12-12 \\ 12+12 & -16+9 \end{bmatrix} = \begin{bmatrix} -7 & -24 \\ 24 & -7 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 3+8 & -4+6 \\ -6+4 & 8+3 \end{bmatrix} = \begin{bmatrix} 11 & 2 \\ -2 & 11 \end{bmatrix}$

$\therefore \quad 2\,AB = \begin{bmatrix} 22 & 4 \\ -4 & 22 \end{bmatrix}$

$\therefore \quad A^2 + B^2 + 2\,AB = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} + \begin{bmatrix} -7 & -24 \\ 24 & -7 \end{bmatrix} + \begin{bmatrix} 22 & 4 \\ -4 & 22 \end{bmatrix}$

$= \begin{bmatrix} -3-7+22 & 4-24+4 \\ -4+24-4 & -3-7+22 \end{bmatrix}$

$\therefore \quad A^2 + B^2 + 2\,AB = \begin{bmatrix} 12 & -16 \\ 16 & 12 \end{bmatrix}$ ...(2)

From (1) and (2), we get, $(A+B)^2 = A^2 + B^2 + 2\,AB$.

Example9:If

$$A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}, \text{ then } A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}$$

Where n is a positive integer.

Sol. Here $A = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$

$$\therefore \quad A^1 = \begin{bmatrix} 1+10\cdot1 & -25\cdot1 \\ 4\cdot1 & 1-10\cdot1 \end{bmatrix}$$

$\therefore$ result in true for $n = 1$

Assume that the result is true for $n = k$

$$\therefore \; A^k = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}$$

$$A^{k+1} = A^k \,.A = \begin{bmatrix} 1+10k & -25k \\ 4k & 1-10k \end{bmatrix}\begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 11+110k-100k & -25-250k+225k \\ 44k+4-40k & -100k-9+90k \end{bmatrix}$$

$$= \begin{bmatrix} 11+10k & -25-25k \\ 4+4k & -9-10k \end{bmatrix}$$

$$\therefore \; A^{k+1} = \begin{bmatrix} 1+10(k+1) & -25(k+1) \\ 4(k+1) & 1-10(k+1) \end{bmatrix}$$

Therefore result is true for n = k+1

If the result is true for n = k, then it is also true for n = k=1 i.e. if the result is true for any integer, then it is also true for the next integer.

But the result is true for integer 1.

Therefore by principle of mathematical induction, the result is true for every positive integer n.

$$\therefore \quad A^n = \begin{bmatrix} 1+10n & -25n \\ 4n & 1-10n \end{bmatrix}$$

where n is any positive integer.

$= 1$

Example 10: Three shopkeepers A, B and C go to a store to buy stationery. A purchase 12 dozen note-books, 5 dozen pens and 6 dozen pencils. B purchase 10 dozen note-books, 6 dozen pens and 7 dozen pencils. C purchase 11 dozen note-books, 13 dozen pens and 8 dozen pencils. A note book costs 40 paise, a pen costs Rs. 1-25 and a pencil costs 35 paise. Use matrix multiplication to calculate each individual's bill.

Sol: A purchase 144 note-books, 60 pens, 72 pencils.
B purchase 120 note-books, 72 pens, 84 pencils.
C purchase 132 note-books, 156 pens, 96 pencils.

Let $D = \begin{bmatrix} 144 & 60 & 72 \\ 120 & 72 & 84 \\ 132 & 156 & 96 \end{bmatrix}$ be the matrix of purchases.

Let E be the price matrix.

$$\therefore \quad E = \begin{bmatrix} 0 \cdot 40 \\ 1 \cdot 25 \\ 0 \cdot 35 \end{bmatrix}$$

$$\therefore \quad DE = \begin{bmatrix} 144 & 60 & 72 \\ 120 & 72 & 84 \\ 132 & 156 & 96 \end{bmatrix} \begin{bmatrix} 0 \cdot 40 \\ 1 \cdot 25 \\ 0 \cdot 35 \end{bmatrix}$$

$$= \begin{bmatrix} 57 \cdot 60 + 75 \cdot 00 + 25 \cdot 20 \\ 48 \cdot 00 + 90 \cdot 00 + 29 \cdot 40 \\ 52 \cdot 80 + 195 \cdot 00 + 33 \cdot 60 \end{bmatrix} = \begin{bmatrix} 157 \cdot 80 \\ 157 \cdot 40 \\ 281 \cdot 40 \end{bmatrix}$$

$\therefore$ bill of A, B, C are Rs. 157·80, Rs. 167·40 and Rs. 281·40 respectively.

4.4.10 Transpose of a Matrix

The matrix obtained from a given matrix A, by interchanging its rows and columns, is called the transpose of A and is generally denoted A' or A' or A^T.

Thus if A = [a i j], then A' = [a ji] where a j i element of A' is the (i,j) th element of A.

For example if $A = \begin{bmatrix} 2 & 4 & 7 \\ 1 & 3 & 5 \end{bmatrix}$, then $A' = \begin{matrix} 2 & 1 \\ 4 & 3 \\ 7 & 5 \end{matrix}$

you may verify easily that (A')' = A.

i.e. transpose of the transpose of any matrix is the same matrix.

Prove the following

(i) (A + B)' = A' + B' i.e., transpose of the sum of. two matrices is sum of the transposes of the matrices.

(ii) (kA)’ = Ka’, k being a complex number.

Verification

(*i*) Let $A = \begin{bmatrix} 2 & 3 & 7 \\ 3 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 & -3 \\ -2 & 5 & 8 \end{bmatrix}$.

$$A + B = \begin{bmatrix} 2 & 3 & 7 \\ 3 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 1 & -3 \\ -2 & 5 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 4 & 4 \\ 1 & 6 & 12 \end{bmatrix}$$

$$(A + B)' = \begin{bmatrix} 7 & 1 \\ 4 & 6 \\ 4 & 12 \end{bmatrix} \qquad \ldots(1)$$

$$A' = \begin{bmatrix} 2 & 3 \\ 3 & 1 \\ 7 & 4 \end{bmatrix},\ B' = \begin{bmatrix} 5 & -2 \\ 1 & 5 \\ -3 & 8 \end{bmatrix}$$

$$A' + B' = \begin{bmatrix} 2 & 3 \\ 3 & 1 \\ 7 & 4 \end{bmatrix} + \begin{bmatrix} 5 & -2 \\ 1 & 5 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 4 & 6 \\ 4 & 12 \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2), we get,

$$(A + B)' = A' + B'$$

(*ii*) Let $k = 3, \ A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$

$$\therefore \quad kA = 3\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 15 & 21 \end{bmatrix}$$

$$\therefore \quad (kA)' = \begin{bmatrix} 6 & 15 \\ 9 & 21 \end{bmatrix} \qquad \ldots(1)$$

Also $kA' = 3\begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 6 & 15 \\ 9 & 21 \end{bmatrix}$ ...(2)

From (1) and (2), we get,

$$(kA)' = kA'.$$

4.4.11 Reversal Law

If A and B are any two matrices conformable for multiplication, then (AB)' = B' A' *i.e.,* the transpose of the product of two matrices is the product of the transposes taken in the reverse order.

Verification

Let $A = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 5 \\ 1 & 8 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 6+3 & 10+24 \\ 15+6 & 25+48 \end{bmatrix} = \begin{bmatrix} 9 & 34 \\ 21 & 73 \end{bmatrix}$$

$$\therefore \quad (AB)' = \begin{bmatrix} 9 & 21 \\ 34 & 73 \end{bmatrix} \qquad \ldots(1)$$

$$A' = \begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix}, \ B' = \begin{bmatrix} 3 & 1 \\ 5 & 8 \end{bmatrix}$$

$$\therefore \quad B'A' = \begin{bmatrix} 3 & 1 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 6+3 & 15+6 \\ 10+24 & 25+48 \end{bmatrix}$$

$$\therefore \quad B'A' = \begin{bmatrix} 9 & 21 \\ 34 & 73 \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2), we get,

$$(AB)' = B'A'.$$

4.4.12 Symmetric and Skew-Symmetric Matrices

(I) Any square matrix A = [a i j] is said to be a symmetric matrix if a i j = a j i i.e., (i, j) th element of A is the same as the (j,i) th element of A. If we take the transpose of a symmetric matrix A, it is the same as A.Therefore, any matrix A is symmetric iff A ' = A.

Example of symmetric matrix are

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 7 & 9 \\ 3 & 7 & 7 & -1 \\ 4 & 9 & -1 & 0 \end{bmatrix}$$

(II) Any square matrix A = [a j j] is said to be a skew-symmetric matrix if a i j = - aj i i.e. (i, j) th element is the same as the negative of the (j, i) element.

∴ For a skew-symmetric matrix A, a i j = - a j i

∴ Putting $j = i$, we get a a i i = - aii or aii = 0 *i.e.,* every diagonal element of A is zero.

Examples of skew-symmetric matrix rate

$$\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}, \begin{bmatrix} 0 & -5 \\ 5 & 0 \end{bmatrix}$$

Note: A matrix A is set to be:

1.Symmetric if A’ = A

2. Skew- symmetric if A’ = -A

Example 1. If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$, then verify that $AA^T = A^T A = I$

Sol. Here $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \Rightarrow A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$

$$\therefore \quad AA^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & -2+4-2 \\ 2+2-4 & 4+1+4 & -4+2+2 \\ -2+4-2 & -4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Also $A^T A = \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & 2-4+2 \\ 2+2-4 & 4+1+4 & 4-2-2 \\ 2-4+2 & 4-2-2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore \quad AA^T = A^T A = I$

Example 2. If $A = \begin{bmatrix} 1 & -4 \\ 0 & 5 \\ 6 & 7 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & -7 \end{bmatrix}$ then verify that $(AB)' = B' A'$.

Sol. Here $A = \begin{bmatrix} 1 & -4 \\ 0 & 5 \\ 6 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & -7 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 1 & -4 \\ 0 & 5 \\ 6 & 7 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 2-4 & 3+0 & -1+28 \\ 0+5 & 0+0 & 0-35 \\ 12+7 & 18+0 & -6-49 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 27 \\ 5 & 0 & -35 \\ 19 & 18 & -55 \end{bmatrix}$$

$$\therefore \quad (AB)' = \begin{bmatrix} -2 & 5 & 19 \\ 3 & 0 & 18 \\ 27 & -35 & -55 \end{bmatrix} \qquad \ldots(1)$$

Again $A' = \begin{bmatrix} 1 & 0 & 6 \\ -4 & 5 & 7 \end{bmatrix}, B' = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & -7 \end{bmatrix}$

$$\therefore \quad B'A' = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & -7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 6 \\ -4 & 5 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2-4 & 0+5 & 12+7 \\ 3+0 & 0+0 & 18+0 \\ -1+28 & 0-35 & -6-49 \end{bmatrix}$$

$$\therefore \quad B'A' = \begin{bmatrix} -2 & 5 & 19 \\ 3 & 0 & 18 \\ 27 & -35 & -55 \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2), we get, $(AB)' = B'A'$.

Example 3: A and B are symmetric. Show that AB is symmetric iff AB = BA.

Sol: $\because$ *A and B are symmetric*

Therefore A' = A...(1)

B' = B

Let AB = BA...(2)

Now (AB)' = B'A' = BA [$\because$ *of* (1)]

Therefore (AB)' = AB [$\because$ **of 2**]

Therefore, AB is symmetric.

Again let AB be symmetric i.e., (AB)' = AB

$\because$ **(AB)' = AB**

therefore B'A' = AB

$\therefore$ **BA = AB, which proves the required result.**

Example 4. Express the matrix

$$A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}.$$

as the sum of a symmetric and skew–symmetric matrix.

Sol. $A = \begin{bmatrix} 4 & 2 & -3 \\ 1 & 3 & -6 \\ -5 & 0 & -7 \end{bmatrix}$, $\therefore$ $A' = \begin{bmatrix} 4 & 1 & -5 \\ 2 & 3 & 0 \\ -3 & -6 & -7 \end{bmatrix}$

$\therefore$ $A + A' = \begin{bmatrix} 8 & 3 & -8 \\ 3 & 6 & -6 \\ -8 & -6 & -14 \end{bmatrix}$, $\therefore$ $\frac{1}{2}(A + A') = \begin{bmatrix} 4 & \frac{3}{2} & -4 \\ \frac{3}{2} & 3 & -3 \\ -4 & -3 & -7 \end{bmatrix}$

which is a symmetric matrix.

Again $\frac{1}{2}(A - A') = \begin{bmatrix} 0 & \frac{1}{2} & 1 \\ -\frac{1}{2} & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$

which is a skew-symmetric matrix.

Now $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

$\therefore$ A has been expressed as the sum of a symmetric and a skew symmetric matrix.

Example 5: Show that every square matrix can be expressed in one and only one way as a sum of a symmetric and a skew symmetric matrix.

Sol. We have $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$ where A is any square matrix.

$\therefore \quad A = P + Q$ where $P = \frac{1}{2}(A + A')$ and $Q = \frac{1}{2}(A - A')$

Now $P' = \frac{1}{2}(A + A')' = \frac{1}{2}[A' + (A')'] = \frac{1}{2}[A' + A] \quad [\because (A')' = A]$

$= \frac{1}{2}(A + A')$

$\therefore \quad P' = P$ *i.e.*, P is symmetric.

Again $Q' = \frac{1}{2}(A - A')' = \frac{1}{2}[A' - (A')'] = \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A')$

$\therefore \quad Q' = -Q$

$\therefore$ Q is a skew-symmetric matrix

$\therefore \quad A = P + Q \qquad \ldots(1)$

Where P and Q are symmetric and skew-symmetric matrices respectively.

Equation (1) shows that every square matrix can be expressed in one way as the sum of a symmetric and a skew-symmetric matrix.

Now we will prove that representation (1) is unique. For this,

Let $A = R+S$ …..(2)

be another representation where R is symmetric and S is skew symmetric.

Now $\quad A' = (R + S)' = R' + S'$

$A' = R - S$

$[\because R' = R,\ S' = -S]$

From (1) and (3), we get,

R = A+A'/2 = P and S = A-A'/2 = Q

$\because$ *the representation* (2), *is the same as representaion* (1). *hence the result*,

Another form of above example :

Let A be a square matrix. Show that $\frac{1}{2}$ (A + A′) is a symmetric matrix and $\frac{1}{2}$ (A – A′) is a skew-symmetric matrix.

Conclude that any square matrix can be written as sum of a symmetric matrix and a skew symmetric matrix.

(Hint. Do not prove that representation is unique).

4.5 ADJOINT AND INVERSE OF MATRICES

4.5.1 Adjoint of a Square Matrix

Let A = [aij] be any square matrix.

The transpose of the matrix [Aij], where Ai j, denotes the co-factors of a_{ij} in | A |, is called the adjoint of A and is denoted by adj. A.
In other words adj. A is a matrix obtained after replacing every element *an* of A = [a_{ij}] by its co-factor A_{ij} in | A | and then taking the transpose of the matrix formed by co-factors.

For example, let A = $\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ $then$ | A |, = $\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$

Co factor of 2 = 7

Cofactor of 3 = -4

Cofactor of 4 = -3

Cofactor of 7 = 2

$$\therefore \quad \text{Adj. A} = \begin{bmatrix} 7 & -4 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} 7 & -3 \\ -4 & 2 \end{bmatrix}$$

Thus if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\text{Adj. A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Note. Adj. A is sometimes called adjugate of A.

Prove that: A (adj. A) = | A | I = (adj A) A ***i.e.*** the matrices A and commutative and their product is a scalar matrix whose every diagonal element is | A |.

Proof. Let A = [aij] be any square matrix of order n**. ,**

Now (*i, j*) th element of A adj. A = product of *i* th row of A and jth column of adj. A

$$= a_{i1} A_{j1} + a_{i2} A_{j2} + \qquad + a_{in} A_{jn}$$

$$= 0 \text{ or } |A| \text{ according as } i \neq j \text{ or } i = j.$$

[in order to obtain the above result, we have applied the following two results of determinants : ,

(i) If the elements of any line of a determinant are multiplied by the co-factors of the corresponding elements of any other parallel line, the sum of the products so formed is zero.

(ii) If the elements of any line of a determinant are multiplied by their own factors, the sum of the products so formed is the determinant itself.]

∴ Every non-diagonal element of A (adj. A) is zero whereas each diagonal element is | A|.

$$\therefore \quad A\,(\text{adj. } A) = \begin{bmatrix} |A| & 0 & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & 0 & \dots & 0 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & \dots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

∴ A (adj. A) = | A | I where I is a unit matrix of order n.

Similarly (adj. A) A = | A | I

∴ we have A (adj. A) = | A | I = (adj. A) A.

Verification:

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 5 \\ 3 & -2 & -4 \\ -3 & 1 & -2 \end{bmatrix}$$

$$\therefore \quad |A| = \begin{vmatrix} 2 & 1 & 5 \\ 3 & -2 & -4 \\ -3 & 1 & -2 \end{vmatrix} = 2\begin{vmatrix} -2 & -4 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 3 & -4 \\ -3 & -2 \end{vmatrix} + 5\begin{vmatrix} 3 & -2 \\ -3 & 1 \end{vmatrix}$$

$$= 2\,(4+4) - 1\,(-6-12) + 5\,(3-6) = 16 + 18 - 15 = 19$$

Co-factor of first element of first row of $|A| = (-1)^{1+1}\begin{vmatrix} -2 & -4 \\ 1 & -2 \end{vmatrix} = 4 + 4 = 8$

Co-factor of second element of first row of $|A| = (-1)^{1+2}\begin{vmatrix} 3 & -4 \\ -3 & -2 \end{vmatrix}$

$= -(-6-12) = 18$

Co-factor of third element of first row of $|A| = (-1)^{1+3}\begin{vmatrix} 3 & -2 \\ -3 & 1 \end{vmatrix}$

$= 3 - 6 = -3$

Co-factor of first element of second row of $|A| = (-1)^{2+1}\begin{vmatrix} 1 & 5 \\ 1 & -2 \end{vmatrix}$

$= -(-2-5) = 7$

Co-factor of second element of second row of $|A| = (-1)^{2+2}\begin{vmatrix} 2 & 5 \\ -3 & -2 \end{vmatrix}$

$= -4 + 15 = 11$

Co-factor of third element of second row of $|A| = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix}$

$= -(2+3) = -5$

Co-factor of first element of third row of $|A| = (-1)^{3+1}\begin{vmatrix} 1 & 5 \\ -2 & -4 \end{vmatrix}$

$= -4 + 10 = 6$

Co-factor of second element of third row of $|A| = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix}$

$= -(-8-15) = 23$

Co-factor of third element of third row of $|A| = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}$

$= -4 - 3 = -7$

$$\therefore \quad \text{adj. } A = \begin{bmatrix} 8 & 18 & -3 \\ 7 & 11 & -5 \\ 6 & 23 & -7 \end{bmatrix}' = \begin{bmatrix} 8 & 7 & 6 \\ 18 & 11 & 23 \\ -3 & -5 & -7 \end{bmatrix}$$

$$A\,(\text{adj } A) = \begin{bmatrix} 2 & 1 & 5 \\ 3 & -2 & -4 \\ -3 & 1 & -2 \end{bmatrix}\begin{bmatrix} 8 & 7 & 6 \\ 18 & 11 & 23 \\ -3 & -5 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 16+18-15 & 14+11-25 & 12+23-35 \\ 24-36+12 & 21-22+20 & 18-46+28 \\ -24+18+6 & -21+11+10 & -18+23+14 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{bmatrix} = 19\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore \quad A\,(\text{adj } A) = 19\,I$

$\Rightarrow \quad A\,(\text{adj } A) = |A|\,I \qquad \ldots(1)$

$$(\text{adj } A)\,A = \begin{bmatrix} 8 & 7 & 6 \\ 18 & 11 & 23 \\ -3 & -5 & -7 \end{bmatrix} \begin{bmatrix} 2 & 1 & 5 \\ 3 & -2 & -4 \\ -3 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 16+21-18 & 8-14+6 & 40-28-12 \\ 36+33-69 & 18-22+23 & 90-44-46 \\ -6-15+21 & -3+10-7 & -15+20+14 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{bmatrix} = 19 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= 19\,I$

$\therefore \quad (\text{adj } A)\,A = |A|\,I \qquad \ldots(2)$

From (1) and (2), we get,

$A\,(\text{adj } A) = (\text{adj } A)\,A = |A|\,I.$

Cor. $\because \quad A\,(\text{adj. } A) = |A|\,I, \quad \therefore \; |A\,(\text{adj. } A)| = |A|^n$

$\therefore \quad |A| \; |\text{adj. } A| = |A|^n, \; \therefore \; |\text{adj. } A| = |A|^{n-1}.$

4.5.2 Inverse of a Matrix

Any n-rowed matrix A is said to be invertible if there exists an *n* rowed matrix B such that AB = BA = I, where I is unit matrix of order *n.* Also B is called an inverse of A.

Note 1. If B is an inverse of A, then A is also an inverse of B.

Inverse of a matrix is also sometimes called reciprocal of a matrix.

Prove that inverse of a matrix is always unique.

Proof: Let A be any invertible matrix. Let B and C be two inverses of A.

B is inverse of A, AB = BA = I

C is an inverse of A

AC = CA = I Now CAB = C (AB) = C (I) = C

CAB = C

Again CAB = (CA)B = IB = B

$\therefore$ CAB= B ………...(4)

From (3) and (4), we get,

B= C

$\therefore$ the inverse of any matrix A is unique.

Prove that any square matrix A is invertible iff $|A| \neq 0$.

Proof. Condition is Necessary.

Assume that A is invertible.

there exists' another square matrix B such that AB = BA = I

$\therefore$ I AB I = I

$\therefore \quad |A||B| = |I| \qquad [\because |AB| = |A||B|]$

$\therefore \quad |A||B| = 1 \qquad [\because |I| = 1]$

$\Rightarrow \quad |A| \neq 0.$

Condition is sufficient

Assume that $|A| \neq 0$

Take $B = \dfrac{\text{adj. } A}{|A|}$

$\therefore \quad AB = A\dfrac{\text{adj. } A}{|A|} = \dfrac{1}{|A|} A(\text{adj. } A) = \dfrac{1}{|A|}|A|I.$

$\therefore \quad AB = I$

Similarly BA = I

$\therefore$ we have AB = BA = I

$\therefore \quad B = \dfrac{\text{adj. } A}{|A|}$ is the inverse of A

$\therefore$ A is invertible and its inverse is $\dfrac{\text{adj. } A}{|A|}$.

4.5.3 Singular and Non-Singular Matrix

Any matrix A is said to be singular if $| A | = 0$ and non-singular if $|A| \neq 0$

'But A is invertible iff $| A | \neq 0$

only non-singular matrices are invertible.

If A, B be two non-singular matrices of the same order, prove that $(AB)^{-1} = B^{-1}A^{-1}$ *i.e.* the inverse of a product of two matrices is the product of the inverses taken in the reverse order.

Proof. We have

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI A^{-1} = AA^{-1} = I$

Similarly $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$.

Therefore,

$\therefore B^{-1}A^{-1}$ is inverse of AB.

$\therefore \quad (AB)\ (B^{-1}A^{-1}) = (B^{-1}A^{-1})\ (AB) = I$

$\therefore \quad B^{-1}A^{-1}$ is inverse of AB

or $\quad (AB)^{-1} = B^{-1}A^{-1}$

Art-38. If A be an n-rowed non-singular matrix, prove that

(*i*) $(A')^{-1} = (A^{-1})'$ where A' is the transpose of A

(*ii*) $(A^k)^{-1} = (A^{-1})^k$ where k is any positive integer.

(*iii*) $(A^\theta)^{-1} = (A^{-1})^\theta$

(*iv*) $(A^{-1})^{-1} = A$

Proof. (*i*) We have

$$AA^{-1} = A^{-1}A = I, \qquad \therefore\ (AA^{-1})' = (A^{-1}A)' = I'$$

$\Rightarrow \quad (A^{-1})'\ A' = A'\ (A^{-1})' = I \qquad\qquad (\because\ I' = I)$

$\Rightarrow \quad (A^{-1})'$ is the inverse of $A' \quad \therefore (A')^{-1} = (A^{-1})'$

(*ii*) $\quad$ L.H.S $= (A^k)^{-1} = (A\ .\ A\ .\ A\ldots\text{to } k \text{ factors})^{-1}$

$\qquad = A^{-1}\ .\ A^{-1}\ .\ A^{-1}\ldots\text{to } k \text{ factors} = (A^{-1})^k$

$\qquad =$ R.H.S.

(*iii*) We have $\quad AA^{-1} = A^{-1}A = I$

$\therefore \quad (AA^{-1})^\theta = (A^{-1}A)^\theta = I^\theta$

$\Rightarrow \quad (A^{-1})^\theta A^\theta = A^\theta (A^{-1})^\theta = I \qquad\qquad [\because\ I^\theta = I]$

$\Rightarrow \quad (A^{-1})^\theta$ is the inverse of $A^\theta \quad \Rightarrow \quad (A^\theta)^{-1} = (A^{-1})^\theta$

(*iv*) We know that $\ AA^{-1} = A^{-1}A = I$

Replacing A by A^{-1}, we get,

$$A^{-1}(A^{-1})^{-1} = (A^{-1})^{-1} A^{-1} = I$$

or $\qquad A^{-1}(A^{-1})^{-1} = I$

$\Rightarrow \quad A\ .\ A^{-1}(A^{-1})^{-1} = A\ .\ I \qquad \Rightarrow \qquad (A\,A^{-1})\,(A^{-1})^{-1} = A$

$\Rightarrow \quad I\,(A^{-1})^{-1} = A \qquad \Rightarrow \qquad (A^{-1})^{-1} = A$

Example 1. Find the adjoint of the matrix $\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$

Sol. Let $\quad A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}, \quad \therefore\ |A| = \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}$

Co-factor of first element of first row $= (-1)^{1+1}(1) = 1$

Co-factor of second element of first row $= (-1)^{1+2}(5) = -5$

Co-factor of first element of second row $= (-1)^{2+1}(3) = -3$
Co-factor of second element of second row $= (-1)^{2+2}(2) = 2$

$$\therefore \quad \text{adj. A} = \begin{bmatrix} 1 & -5 \\ -3 & 2 \end{bmatrix}' = \begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$$

Example 2. Find the adjoint of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{vmatrix}$$

Co-factors of the elements of first row of | A | are

$$\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix}$$

i.e. $3 - 0,\ -(2 + 10),\ 0 + 6$ *i.e.* $3,\ -12,\ 6$ respectively.

Co-factors of the elements of second row of | A | are

$$-\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix}$$

i.e. $-(-1-0),\ 1+4,\ -(0-2)$ *i.e.* $1, 5, 2$ respectively.

Cofactors of the elements of third row of | A | are

$$\begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$$

i.e. $-5-6,\ -(5-4),\ 3+2 \quad \therefore \quad -11,\ -1,\ 5$ respectively.

$$\therefore \quad \text{adj. A} = \begin{bmatrix} 3 & -12 & 6 \\ 1 & 5 & 2 \\ -11 & -1 & 5 \end{bmatrix}' = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$$

Example 3. Find the sum of $\begin{bmatrix} 2 & -3 \\ 5 & -7 \end{bmatrix}$ and its multiplication inverse.

$A + A^t$

A's multiplicative inverse

Sol. Let $A = \begin{bmatrix} 2 & -3 \\ 5 & -7 \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} 2 & -3 \\ 5 & -7 \end{vmatrix} = -14 + 15 = 1 \neq 0$

$\therefore \quad A^{-1}$ exists.

Co-factors of the elements of first row of | A | are – 7, – 5 respectively.
Co-factors of the elements of second row of | A | are 3, 2 respectively.

$$\therefore \quad \text{adj. } A = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}' = \begin{bmatrix} -7 & 3 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{1}\begin{bmatrix} -7 & 3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ -5 & 2 \end{bmatrix}$$

$$\therefore \quad A + A^{-1} = \begin{bmatrix} 2 & -3 \\ 5 & -7 \end{bmatrix} + \begin{bmatrix} -7 & 3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 2-7 & -3+3 \\ 5-5 & -7+2 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

Example 4. For what value of 'a' is the matrix $\begin{bmatrix} 4 & -3 & -1 \\ 2 & a & 6 \\ 3 & -5 & -4 \end{bmatrix}$ singular ? $|A| = 0.$

Sol. Let $\quad A = \begin{bmatrix} 4 & -3 & -1 \\ 2 & a & 6 \\ 3 & -5 & -4 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 4 & -3 & -1 \\ 2 & a & 6 \\ 3 & -5 & -4 \end{vmatrix}$$

$$= 4\begin{vmatrix} a & 6 \\ -5 & -4 \end{vmatrix} - (-3)\begin{vmatrix} 2 & 6 \\ 3 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 2 & a \\ 3 & -5 \end{vmatrix}$$

$$= 4(-4a + 30) + 3(-8 - 18) - (-10 - 3a)$$

$$= -16a + 120 - 78 + 10 + 3a$$

$$= -13a + 52$$

For A be to be singular, we have,

$|A| = 0 \quad \Rightarrow \quad -13a + 52 = 0 \quad \Rightarrow \quad 13a = 52$

$\Rightarrow \quad a = 4$

Example 5. Verify that $(AB)^{-1} = B^{-1} A^{-1}$ for the matrices A and B where

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}.$$

Sol. $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 6 - 5 = 1$$

Co-factors of elements of first row of $|A|$ are 3 and -5 respectively.
Co-factors of elements of second row of $|A|$ are -1 and 2 respectively.

$$\text{adj. } A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj. } A}{|A|} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} = 16 - 15 = 1$$

Co-factors of elements of first row of $|B|$ are 4 and -3 respectively.
Co-factors of element of second row of $|B|$ are -5 and 4 respectively.

$$\text{adj.} B = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}' = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$$

$$B^{-1} = \frac{\text{adj. } B}{|B|} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$$

$$\therefore \quad B^{-1} A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12 + 25 & -4 - 10 \\ -9 - 20 & 3 + 8 \end{bmatrix}$$

$$\therefore \quad B^{-1} A^{-1} = \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix} \quad \ldots(1)$$

$$AB = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ 29 & 37 \end{bmatrix}$$

$$|AB| = \begin{bmatrix} 11 & 14 \\ 29 & 37 \end{bmatrix} = 407 - 406 = 1$$

Co-factors of the first row of | AB | are 37, – 29 respectively.
Co-factors of the second row of | AB | are – 14, 11 respectively.

$$\text{adj. (AB)} = \begin{bmatrix} 37 & -29 \\ -14 & 11 \end{bmatrix}' = \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$$

$$\therefore \qquad (AB)^{-1} = \frac{\text{adj. (AB)}}{|\,AB\,|}$$

$$(AB)^{-1} = \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix} \qquad \ldots(2)$$

From (1) and (2), we have,

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Example 6. Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$.

Sol. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$, $\therefore \quad |A| = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{vmatrix}$

$$= 1\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix}$$

$$= 1\,(8 - 6) + 1\,(0 + 9) + 2\,(0 - 6)$$

$$= 2 + 9 + 12 = -1 \neq 0.$$

$\therefore \quad A^{-1}$ exists

Co-factors of the elements of first row of | A | are

$$\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix}, -\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix}$$

i.e. 2, – 9, – 6 respectively.

Co-factors of the elements of second row of | A | are

$$-\begin{vmatrix} -1 & 2 \\ -2 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ 3 & -2 \end{vmatrix}$$

i.e. 0, – 2, – 1 respectively.

Co-factors of the elements of third row of | A | are

$$\begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix}, -\begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}$$

i.e. – 1, 3, 2 respectively.

$$\therefore \quad \text{adj } A = \begin{bmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj. } A}{|A|} = \frac{1}{-1} \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Example 7. If $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, prove that $A^{-1} = A^2 - 6A + 11I$

Sol. Here $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$$\therefore |A| = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + (-1) \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 2(3-0) - 0 - (5-0) = 6 - 5 = 1 \neq 0$$

$\therefore \quad A^{-1}$ exists

Co-factors of the elements of first row of $|A|$ are

$$\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix}, -\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix}, \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix}$$

i.e. 3, – 15, 5 respectively

Co-factors of the elements of second row of $|A|$ are

$$-\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}, -\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}$$

i.e. – 1, 6, – 2 respectively

Co-factors of the elements of third row of $|A|$ are

$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 2 & -1 \\ 5 & 0 \end{vmatrix}, \begin{vmatrix} 2 & 0 \\ 5 & 1 \end{vmatrix}$$

i.e. 1, – 5, 2 respectively

$$\therefore \quad \text{adj. A} = \begin{bmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj. A}}{|A|} = \frac{1}{1}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+0 & 0+0-1 & -2+0-3 \\ 10+5+0 & 0+1+0 & -5+0+0 \\ 0+5+0 & 0+1+3 & 0+0+9 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix}$$

$$\text{R.H.S.} = A^2 - 6A + 11I$$

$$= \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix} - 6\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 & -5 \\ 15 & 1 & -5 \\ 5 & 4 & 9 \end{bmatrix} + \begin{bmatrix} -12 & 0 & 6 \\ -30 & -6 & 0 \\ 0 & -6 & -18 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 4-12+11 & -1+0+0 & -5+6+0 \\ 15-30+0 & 1-6+11 & -5+0+0 \\ 5+0+0 & 4-6+0 & 9-18+11 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} = A^2 = \text{L.H.S.}$$

4.6 CONCEPT OF DETERMINANT

We know that a matrix is an arrangement of numbers and not a number. With every square matrix A = [aij], we associate a number called the determinant of A and denoted by det A or | A |.

Let A = [a i j] be a square matrix of order n. Then the number:

$$\left|\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\right| \text{ or } \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

is associated to the square matrix A and is called determinant of the matrix A and is denoted by | A | or det A.

The determinant of a 1 **X** 1 matrix A = [a_{11}] is defined to be a_{11}.
Therefore, [a11] = [-7], then A = -7.

4.7 DETERMINANT OF A SQUARE MATRIX

4.7.1 Determinant of 2nd Order

The symbol $\begin{vmatrix} a1 & b1 \\ a2 & b2 \end{vmatrix}$ in which any four number a_1, b_2, a_2, b_2 are arranged in square array consisting of two horizontal lines and two vertical lines, and bounded by two vertical bars is called a determinant of 2nd order. Each horizontal line is called a row and each vertical line is called a column. The four numbers are called elements. The elements a1, b2 in the above arrangement are said to form the leading or principal diagonal.

4.7.2 Determinant of Third Order

The symbol $\begin{vmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{vmatrix}$ in which nine numbers a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3 are arranged in two vertical bars, is called a determinant of third order.

4.8 EXPANSION RULE

4.8.1 Expansion of Determinant of 2nd order

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Now to expand Δ, we multiply the elements of leading **diagonal** and from this product, we subtract the product of the elements of the other diagonal.

$$\therefore \quad \Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

4.8.2 Expansion of Determinant of Third Order

Let us expand a given determinant A by the first row. We multiply the elements of first row by their corresponding cofactors and add the products, the sum so obtained gives the required solution.

1

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\text{Co-factor of } a_1 = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$\text{Co-factor of } b_1 = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$\text{Co-factor of } c_1 = (-1)^{1+3} \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$\therefore$ by above rule of expansion, we have,

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$\therefore \Delta a_1 1 A_1 + b_1 1 + B_1 + c_1 1 C_1$

Similarly Δ a2 A2 + b2B2 + c2 C2

And $\Delta = a3A3 + b3B3 + c3C3$

Also Δ = a1A1 + a2A2 + a3A3 = b1B1 + b2B2 + b3B3

= c1C1 + c2C2 + c3C3

Thus, we have the property,

the sum of the product of elements of a row of given determinant A with their corresponding co-factors is the value of A.

Similarly we can prove that the sum of the products of elements of any row with the co-factors of the corresponding elements of some other row is zero.

4.8.3 Sarus Rule

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Write the three columns and repeat columns 1 and 2.

Draw arrows as shown in the figure given below.

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

The value of the determinant Δ is obtained by adding all the products on the downward moving arrows and subtracting the products on the upward moving arrows.

For example, if

$$\Delta = \begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

$$\begin{array}{ccccc} 6 & -3 & 2 & 6 & -3 \\ 2 & -1 & 2 & 2 & -1 \\ -10 & 5 & 2 & -10 & 5 \end{array}$$

$$\therefore \quad \Delta = [\,(6)(-1)(2) + (-3)(2)(-10) + (2)(2)(5)\,] - [\,(-10)(-1)(2) + (5)(2)(6) + (2)(2)(-3)\,]$$

$$= (-12 + 60 + 20) - (20 + 60 - 12) = 68 - 68 = 0$$

ILLUSTRATIVE EXAMPLES:

Example 1. Find the value of Δ, where

$$\Delta = \begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}.$$

Sol. Here $\Delta = \begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$

$$= 6\begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 2 & 2 \\ -10 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -10 & 5 \end{vmatrix}$$

$$= 6(-2-10) + 3(4+20) + 2(10-10) = -72 + 72 = 0.$$

Example 2. Write the co-factors of the elements of the second row and hence evaluate the determinant $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$

Sol. Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$

Co-factor of first element of 2nd row $= (-1)^{2+1}\begin{vmatrix} 2 & 3 \\ -7 & 9 \end{vmatrix}$

$$= -\begin{vmatrix} 2 & 3 \\ -7 & 9 \end{vmatrix} = -(18+21) = -39$$

Co-factor of 2nd element of 2nd row $= (-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 2 & 9 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 3 \\ 2 & 9 \end{vmatrix} = 9 - 6 = 3$$

Co-factor of 3rd element of 2nd row $= (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & -7 \end{vmatrix}$

$$= -\begin{vmatrix} 1 & 2 \\ 2 & -7 \end{vmatrix} = -(-7-4) = 11$$

$\therefore \quad \Delta = (-4)(-39) + (3)(3) + (6)(11) = 156 + 9 + 66 = 231.$

Example 3. Show that for the matrix

$$A = \begin{bmatrix} 5 & 7 & 8 \\ 6 & 4 & 3 \\ 1 & 2 & 5 \end{bmatrix}; \ |A| = |A'| \text{ where } A' \text{ denotes the transpose of } A.$$

Sol. $$A = \begin{bmatrix} 5 & 7 & 8 \\ 6 & 4 & 3 \\ 1 & 2 & 5 \end{bmatrix}$$

$$\Rightarrow \quad |A| = \begin{vmatrix} 5 & 7 & 8 \\ 6 & 4 & 3 \\ 1 & 2 & 5 \end{vmatrix} = 5\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix} - 7\begin{vmatrix} 6 & 3 \\ 1 & 5 \end{vmatrix} + 8\begin{vmatrix} 6 & 4 \\ 1 & 2 \end{vmatrix}$$

$$= 5\,(20-6) - 7\,(30-3) + 8\,(12-4) = 70 - 189 + 64 = 134 - 189$$
$$= -55$$

Again $$A' = \begin{bmatrix} 5 & 6 & 1 \\ 7 & 4 & 2 \\ 8 & 3 & 5 \end{bmatrix}$$

$$\Rightarrow \quad |A'| = \begin{vmatrix} 5 & 6 & 1 \\ 7 & 4 & 2 \\ 8 & 3 & 5 \end{vmatrix} = 5\begin{vmatrix} 4 & 2 \\ 3 & 5 \end{vmatrix} - 6\begin{vmatrix} 7 & 2 \\ 8 & 5 \end{vmatrix} + 1\begin{vmatrix} 7 & 4 \\ 8 & 3 \end{vmatrix}$$

$$= 5\,(20-6) - 6\,(35-16) + 1\,(21-32)$$
$$= 70 - 114 - 11 = -55$$

$$\therefore \quad |A| = |A'|.$$

Example 4. Evaluate $\begin{vmatrix} 5 & 2 & 1 \\ 3 & 0 & 2 \\ 8 & 1 & 3 \end{vmatrix}$ by two methods.

Sol. Let $$\Delta = \begin{vmatrix} 5 & 2 & 1 \\ 3 & 0 & 2 \\ 8 & 1 & 3 \end{vmatrix}$$

$$= 5\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 8 & 3 \end{vmatrix} + 1\begin{vmatrix} 3 & 0 \\ 8 & 1 \end{vmatrix}$$

$$= 5\,(0-2) - 2\,(9-16) + 1\,(3-0) = -10 + 14 + 3 = 7$$

4.9 PROPERTIES OF DETERMINANT

4.9.1 Property 1. If each element of a row of determinant is zero, then the value of the determinant is zero.

Proof. Let $\Delta = \begin{vmatrix} 0 & 0 & 0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

$$= 0\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - 0\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} + 0\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

(Expanding by First Row)

$= 0 - 0 + 0 = 0.$

4.9.2 Property 2. Value of a determinant is not changed by changing the rows is not columns and columns into rows.

Proof. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\therefore \quad \Delta = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

(Expanding by First Row)

$\therefore \quad \Delta = a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2) \quad \ldots(1)$

Interchange rows of Δ into columns and columns into rows. Let the new determinant be denoted by Δ'.

$$\therefore \quad \Delta' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

(Expanding by First column)

$\therefore \quad \Delta = a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$

[$\because$ of (1)]

$\therefore \quad \Delta' = \Delta$

Hence the result.

Note 1: Since interchange of rows and columns does not change the value of the determinat, whatever we prove for rows remains true for columns.

Note 2: If A is square matrix, then

det A = det. A'

4.9.3 Property 3. If two adjacent rows of a determinant are interchanged, then the sign of the determinant is changed but its numerical value is unchanged.

Proof. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\therefore\ \Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

(Expanding by First Row)

$$\therefore \Delta = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \quad \ldots(1)$$

Interchange 1st and 2nd rows of Δ. Let the new determinant obtained be denoted by Δ'.

$$\therefore \quad \Delta' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding by 2nd row,

$$= -a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} - c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$

$$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$

$$\therefore \quad \Delta' = -\Delta \qquad [\because \text{ of } (1)]$$

Hence the result.

Cor. If any row of a determinant Δ is passed over n rows then the resulting determinant Δ' is given by

$$\Delta' = (-1)^n \Delta.$$

Note. If two rows of a determinant are interchanged, then the sign of the determinant is changed.

4.9.4 Property 4. If two rows of the determinant are identical, then the value of the determinant is zero.

Proof. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

Interchange 1st and 2nd rows of Δ.

Let new determinant is denoted by Δ'.

$$\therefore \quad \Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$\therefore \quad \Delta' = \Delta \qquad \ldots(1)$$

$\because$ Δ' is obtained from Δ by interchanging 1st and 2nd row.

$$\therefore \quad \Delta' = -\Delta \qquad \ldots(2)$$

Subtracting (2) from (1),

$$0 = 2\Delta$$

$$\therefore \quad \Delta = 0.$$

4.9.5 Property 5. If every element of a row is multiplies by the same constant k, then the value of the determinant is multiplied by the constant k.

Proof. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Expanding by 1st row,

$$\Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$\therefore\ \Delta = a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$ ...(1)

Multiplying elements of first row of Δ by k. Let the new determinant be denoted by Δ'.

$$\therefore \quad \Delta' = \begin{vmatrix} k\,a_1 & k\,b_1 & k\,c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= k\,a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - k\,b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + k\,c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\therefore \quad \Delta = k\,a_1 (b_2 c_3 - b_3 c_2) - k\,b_1 (a_2 c_3 - a_3 c_2) + k\,c_1 (a_2 b_3 - a_3 b_2)$$

$$= k \left[a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)\right]$$

$$\therefore \quad \Delta' = k\,\Delta$$

Note. If A is any square matrix of order n then

$$|k\,A| = k^n\,|A|$$

Cor 1. If two rows of a determinant are proportional, then its value is zero.

Proof. Let $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ k\,b_1 & k\,b_2 & k\,b_3 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= k \,.\, 0$ [$\because$ two rows are identical]

$= 0$

Cor. 2 If C_{ij} denoted the cofactor of a_{ij} in $A = [\, a_{ij} \,]$, then

$$\sum_{k=1}^{n} a_{ik}\, C_{jk} = \begin{cases} |A| & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

We have already proved it.

4.9.6 Property 6. If each element in any row consists of two terms, then the determinant can be expressed as the sum of the two determinants of the same order.

Proof. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\therefore \qquad \Delta = a_1 A_1 + b_1 B_1 + c_1 C_1$, where

A_1, B_1, C_1, are co-factors of a_1, b_1, c_1 respectively in Δ.

Let $\Delta' = \begin{vmatrix} a_1 + \alpha_1 & b_1 + \alpha_2 & c_1 + \alpha_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= (a_1 + \alpha_2)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - (b_1 + \alpha_2)\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (c_1 + \alpha_3)\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= (a_1 + \alpha_1) A_1 + (b_1 + \alpha_2) B_1 + (c_1 + \alpha_3) C_1$$

$$= (a_1 A_1 + b_1 B_1 + c_1 C_1) + (\alpha_1 A_1 + \alpha_2 B_1 + \alpha_3 C_1)$$

$$\therefore \qquad \Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & b_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & b_3 \end{vmatrix}$$

Hence the result.

4.9.7 Property 7. The value of a determinant remains unchanged if to each element of a row be added (or subtracted) equimultiplies of the corresponding elements of one or more rows of the determinant.

Proof. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Add l times of 2nd row and m times of 3rd row in first row. Let the new determinant be Δ'. Then

$$\Delta' = \begin{vmatrix} a_1 + l\,a_2 + m\,a_3 & b_1 + l\,b_2 + m\,b_3 & c_1 + l\,c_2 + m\,c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Since each element of first row of Δ' consists of sum of three terms, therefore Δ' can be expressed as the sum of three determinants of the same order.

$$\therefore \quad \Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} l\,a_2 & l\,b_2 & l\,c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m\,a_3 & m\,b_3 & m\,c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + l \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \Delta + l\,.\,0 + m\,.\,0$$

[$\because$ value of a determinant is zero when the rows are identical]

$\therefore \quad \Delta' = \Delta$

Note. All the above properties hold for columns also.

4.9.8 Property 8. Prove that the value of the determinant of a diagonal matrix is equal to the product of the diagonal elements.

4.9.9 Property 9. Prove that the value of the determinant of a skew-symmetric matrix of odd order is always zero.

Proof. Let A be skew symmetric matrix of order n.

$\therefore \quad A' = -A$

$\Rightarrow \quad |A'| = |-A| \quad \Rightarrow \quad |A| = (-1)^n |A|$

$\Rightarrow \quad |A| = -|A| \qquad\qquad [\because\ n \text{ is odd}]$

$\Rightarrow \quad 2|A| = 0 \qquad \Rightarrow \quad |A| = 0$

4.9.10 Property 10. Determinant of a symmetric matrix of even order is always a perfect square.

If the elements of the determinant of a matrix A are rational integral functions of X and two of its rows (or columns) become identical for $X = A$, then $X = A$ is a factor of | A |.

Proof. Since elements of A are rational integral functions of x

$\therefore \quad |A| = f(x).$

Now two rows (or columns) of | A | become identical for $x = a$

$\therefore \quad |A|$ vanishes for $x = a$

$\Rightarrow \quad f(x)$ vanishes for $x = a$

$\therefore$ by factor theorem $(x - a)$ is a factor of $f(x)$.

$\Rightarrow \quad (x - a)$ is a factor of $|A|$.

Cor. If m rows (or columns) of the determinant of a matrix A become identical for $x = a$, then $(x - a)^{m-1}$ is a factor of $|A|$.

Notations:

Let Δ be the given determinant. Then (*i*) R_1 , R_2 , R_3 stand for first, second and third rows of Δ.

(*ii*) C_1 , C_2 , C_3 stand for fisrt, second and third columns of Δ.

(*iii*) By R_2–R_3, we mean that third row is to be subtracted from 2nd row.

(*iv*) By $C_1 + 2\,C_2 - 3\,C_3$ we mean that we are to add in first column, the two times of C_2 and subtract three times of C_3.

ILLUSTRATIVE EXAMPLES;

Example 1. Without expanding, prove that

$$\begin{vmatrix} 9 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix} = 0.$$

Sol. Let $\Delta = \begin{vmatrix} 9 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix} = \dfrac{1}{3}\begin{vmatrix} 9 & 9 & 36 \\ 1 & -3 & -12 \\ 1 & 9 & 36 \end{vmatrix}$

multiply C_3 by 3

$$= \frac{1}{3}\begin{vmatrix} 9 & 9 & 0 \\ 1 & -3 & 0 \\ 1 & 9 & 0 \end{vmatrix}, \text{ by } C_3 - 4\,C_2$$

$= 0$ [$\because$ each element of third column is zero]

Example 2. Without expansion, show that

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0.$$

Sol. Let $\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = (-1)^3 \begin{vmatrix} 0 & -a & b \\ a & 0 & -c \\ -b & c & 0 \end{vmatrix}$,

by taking -1 common from each of three rows

$$= -\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix}, \text{ by interchanging rows and columns.}$$

$$= -\Delta$$

$\therefore \quad \Delta = -\Delta \qquad \Rightarrow \qquad 2\Delta = 0$

$\Rightarrow \quad \Delta = 0$

$$\Rightarrow \begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$$

Note: From the above example it is clear that value of the determinant of a skew-symmetric matrix of third order is always zero.

Example 3: Prove that

at $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0.$

Sol. Let $\Delta = \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & -bc \\ 1 & b & -ca \\ 1 & c & -ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc} \begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc} \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= 0$$

Example 4. Without expanding the determinant, show that $(a + b + c)$ is a factor of the following determinant :

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

Sol. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix}$,

by $R_1 + R_2 + R_3$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix},$$

by taking $(a+b+c)$ common from R_1

$\therefore \quad a+b+c$ is a factor of $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$$= a^2 b^2 c^2 \ (-1) \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = a^2 b^2 c^2 \ (-1)(0-4)$$

$$= 4 a^2 b^2 c^2$$

Example 6. Show that $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} = (x+y+z)^3.$

Sol. Let $\Delta = \begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$

$$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} \quad \text{by} \quad R_1 + R_2 + R_3$$

Taking out $x+y+z$ common from R_1,

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 1 & 0 & 0 \\ 2y & -x-y-z & 0 \\ 2z & 0 & -x-y-z \end{vmatrix},$$

by $C_2 - C_1$, $C_3 - C_1$

$$= (x+y+z)\begin{vmatrix} -x-y-z & 0 \\ 0 & -x-y-z \end{vmatrix},$$

expanding by first row

$$= (x+y+z)\,[(-x-y-z)(-x-y-z) - 0\,.\,0]$$
$$= (x+y+z)(x+y+z)(x+y+z) = (x+y+z)^3.$$

Example 7. Show that

$$\begin{vmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2-c^2) \end{vmatrix} = abc\,(a^2+b^2+c^2)^3$$

Sol. Let

$$\Delta = \begin{vmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2-c^2) \end{vmatrix}$$

$$= a\,b\,c \begin{vmatrix} -b^2 - c^2 + a^2 & 2b^2 & 2c^2 \\ 2a^2 & -c^2 - a^2 + b^2 & 2c^2 \\ 2a^2 & 2b^2 & -a^2 - b^2 + c^2 \end{vmatrix},$$

by taking a, b, c common from C_1, C_2, C_3

$$= a\,b\,c \begin{vmatrix} a^2 + b^2 + c^2 & 2b^2 & 2c^2 \\ a^2 + b^2 + c^2 & -c^2 - a^2 + b^2 & 2c^2 \\ a^2 + b^2 + c^2 & 2b^2 & -a^2 - b^2 + c^2 \end{vmatrix},$$

by $C_1 + C_2 + C_3$

$$= a\,b\,c\,(a^2 + b^2 + c^2) \begin{vmatrix} 1 & 2b^2 & 2c^2 \\ 1 & -c^2 - a^2 + b^2 & 2c^2 \\ 1 & 2b^2 & -a^2 - b^2 + c^2 \end{vmatrix}$$

$$= a\,b\,c\,(a^2 + b^2 + c^2) \begin{vmatrix} 1 & 2b^2 & 2c^2 \\ 0 & -(a^2 + b^2 + c^2) & 0 \\ 0 & 0 & -(a^2 + b^2 + c^2) \end{vmatrix}$$

$$= a\,b\,c\,(a^2 + b^2 + c^2)\,[\,1.\,\{-(a^2 + b^2 + c^2)\}\,.\,\{-(a^2 + b^2 + c^2)\}\,]$$

[**Product of diagonal elements**]

$$= a\,b\,c\,(a^2 + b^2 + c^2)^3$$

Example 8. Prove that

$$\begin{vmatrix} a + b & b + c & c + a \\ b + c & c + a & a + b \\ c + a & a + b & b + c \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Sol. Let $\Delta = \begin{vmatrix} a + b & b + c & c + a \\ b + c & c + a & a + b \\ c + a & a + b & b + c \end{vmatrix}$

$$= \begin{vmatrix} a & b + c & c + a \\ b & c + a & a + b \\ c & a + b & b + c \end{vmatrix} + \begin{vmatrix} b & b + c & c + a \\ c & c + a & a + b \\ a & a + b & b + c \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c+a \\ b & c & a+b \\ c & a & b+c \end{vmatrix} + \begin{vmatrix} a & c & c+a \\ b & a & a+b \\ c & b & b+c \end{vmatrix}$$

$$+ \begin{vmatrix} b & b & c+a \\ c & c & a+b \\ a & a & b+c \end{vmatrix} + \begin{vmatrix} b & c & c+a \\ c & a & a+b \\ a & b & b+c \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} a & b & a \\ b & c & b \\ c & a & c \end{vmatrix} + \begin{vmatrix} a & c & c \\ b & a & a \\ c & b & b \end{vmatrix}$$

$$+ \begin{vmatrix} a & c & a \\ b & a & b \\ c & b & c \end{vmatrix} + 0 + \begin{vmatrix} b & c & c \\ c & a & a \\ a & b & b \end{vmatrix} + \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + 0 + 0 + 0 + 0 + 0 + + \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}.$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= 2 \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

Example 9. Factorise the determinant $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

Sol. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}, \text{ by } C_2 - C_1,\ C_3 - C_1$$

$$= \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}, \text{ expanding by first row}$$

$$= \begin{vmatrix} b-a & c-a \\ (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$$

Taking out $(b-a)$, $(c-a)$ common from C_1 and C_2.

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$$

$$= (b-a)(c-a)\left[(c+a)-(b+a)\right] = (b-a)(c-a)[c-b]$$

$$= (a-b)(b-c)(c-a).$$

Example 10. Prove that :

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Sol. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix},$$

by $C_2 - C_1$, $C_3 - C_1$

$$= \begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}, \text{ expanding with first row.}$$

$$= \begin{vmatrix} b-a & c-a \\ (b-a)(b^2+a^2+ba) & (c-a)(c^2+a^2+ca) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2+a^2+ba & c^2+a^2+ca \end{vmatrix}$$

$$= (b-a)(c-a)\left[c^2+a^2+ca-b^2-a^2-ba\right]$$

$$= (b-a)(c-a)\left[(c^2-b^2)+(c\,a-b\,a)\right]$$

$$= (b-a)(c-a)\left[(c-b)(c+b)+a(c-b)\right]$$

$$= (b-a)(c-a)(c-b)(c+b+a)$$

$$= (a-b)(b-c)(c-a)(a+b+c).$$

Example 11. Prove that

$$\begin{array}{ccc} \downarrow a & \downarrow b & \downarrow c \end{array}$$

$$\begin{vmatrix} a^2+1 & a\,b & a\,c \\ a\,b & b^2+1 & b\,c \\ a\,c & b\,c & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Sol. Let $\Delta = \begin{vmatrix} a^2+1 & a\,b & a\,c \\ a\,b & b^2+1 & b\,c \\ a\,c & b\,c & c^2+1 \end{vmatrix}$

$$= \frac{1}{a\,b\,c}\begin{vmatrix} a(a^2+1) & a\,b^2 & a\,c^2 \\ a^2\,b & b(b^2+1) & b\,c^2 \\ a^2\,c & b^2\,c & c(c^2+1) \end{vmatrix},$$

by multiplying C_1, C_2, C_3 by a, b, c respectively.

$$= \frac{a\,b\,c}{a\,b\,c}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix},$$

by taking a, b, c common from R_1, R_2, R_3 respectively.

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}, \text{ by } C_1+C_2+C_3$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1 + a^2 + b^2 + c^2) \begin{vmatrix} 1 & b^2 & c^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \text{by } R_2 - R_1 , R_3 - R_1$$

$= (1 + a^2 + b^2 + c^2) [1 . 1. 1]$ [Product of diagonal elements]
$= 1 + a^2 + b^2 + c^2.$

Example 12. If x , y , z are different and

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0, \text{ then show that } 1 + x y z = 0$$

Sol. $$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + x y z \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (1 + x y z) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$= (1 + x y z) (x - y) (y - z) (z - x)$

Now $\Delta = 0$

$\Rightarrow$ $(1 + x y z) (x - y) (y - z) (z - x) = 0$

$\Rightarrow$ $1 + x y z = 0$ $\left[\begin{array}{l} \because x , y , z \text{ are all different} \\ \because x - y \neq 0,\ y - z \neq 0,\ z - x \neq 0 \end{array} \right]$

Example 13. Prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2 a b c (a + b + c)^2$$

Sol. Let $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$ ...(1)

Putting $a = 0$ in (1), we get,

$$\Delta = \begin{vmatrix} (b+c)^2 & 0 & 0 \\ b^2 & c^2 & b^2 \\ c^2 & c^2 & b^2 \end{vmatrix} = b^2 c^2 \begin{vmatrix} (b+c)^2 & 0 & 0 \\ b^2 & 1 & 1 \\ c^2 & 1 & 1 \end{vmatrix}$$

$= b^2 c^2 (0) = 0$ [$\because$ two columns are identical]

$\therefore$ $a - 0$ *i.e.*, a is factor of Δ.

Similarly b, c are factors of Δ.

Again putting $a + b + c = 0$ in (1), we get,

$$\Delta = \begin{vmatrix} (-a)^2 & a^2 & a^2 \\ b^2 & (-b)^2 & b^2 \\ c^2 & c^2 & (-c)^2 \end{vmatrix} = \begin{vmatrix} a^2 & a^2 & a^2 \\ b^2 & b^2 & b^2 \\ c^2 & c^2 & c^2 \end{vmatrix} = 0$$

$\therefore$ **$(a + b + c)^2$ is a factor of Δ.**

[$\because$ all the three columns of Δ become identical when we put $a + b + c = 0$]

Now Δ is of sixth degree

$\therefore$ **Δ has got one more linear factor of the type $k(a + b + c)$**

$\therefore$ $\Delta = k\,a\,b\,c\,(a + b + c)^3$ **...(2)**

or $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = k\,a\,b\,c\,(a+b+c)^3$

Put $a = b = c = 1$

$\therefore$ $\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 27k$

$\therefore$ $\begin{vmatrix} 0 & 0 & 1 \\ -3 & 3 & 1 \\ -15 & -3 & 4 \end{vmatrix} = 27k$, by $C_1 - 4C_3$, $C_2 - C_1$

$$\therefore \quad \begin{vmatrix} -3 & 3 \\ -15 & -3 \end{vmatrix} = 27k$$

$$\therefore \quad 9 + 45 = 27k \Rightarrow 27k = 54 \Rightarrow k = 2$$

$$\therefore \quad \text{from (2)}, \Delta = 2abc(a+b+c)^3$$

or
$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

4.10 SOLUTION OF LINEAR EQUATION BY USING CRAMER'S RULE

Cramer's rule. Explain the method of solving the equations

$a_1x + b_1y + c_1z = d1$; $a_2x + b_2y + c_2z = d_2$, and $a_3x + b_3y + c_3z = d_3$

Proof. The given equations are

$$a_1 x + b_1 y + c_1 z = d_1 \quad \ldots(1)$$
$$a_2 x + b_2 y + c_2 z = d_2 \quad \ldots(2)$$
and
$$a_3 x + b_3 y + c_3 z = d_3 \quad \ldots(3)$$

$$\text{Let } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let A_1, A_2, A_3, be the co-factors of a_1, a_2, a_3 respectively in Δ.

Multiplying (1) by A_1, (2) by A_2, (3) by A_3, and adding, we get,

$$(a_1 A_1 + a_2 A_2 + a_3 A_3)\, x + (b_1 A_1 + b_2 A_2 + b_3 A_3)\, y + (c_1 A_1 + c_2 A_2 + c_3 A_3)\, z = d_1 A_1 + d_2 A_2 + d_3 A_3$$

$$\therefore \quad \Delta x + 0.\, y + 0.\, z = d_1 A_1 + d_2 A_2 + d_3 A_3$$

$$\therefore \quad x = \frac{d_1 A_1 + d_2 A_2 + d_3 A_3}{\Delta}$$

$$\therefore \quad x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Similarly, we have

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \text{ and } z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$\therefore$ solution is $x = \dfrac{\Delta_1}{\Delta}, y = \dfrac{\Delta_2}{\Delta}, z = \dfrac{\Delta_3}{\Delta}$ where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

ILLUSTRATIVE EXAMPLES:

Example 1: **Solve the following equations :**

$$3x + y = 7$$
$$5x + 3y = 12.$$

Sol. The given equations are

$$3x + y = 7$$
$$5x + 3y = 12$$

Let $\Delta = \begin{vmatrix} 3 & 1 \\ 5 & 3 \end{vmatrix} = 9 - 5 = 4$

$$\Delta_1 = \begin{vmatrix} 7 & 1 \\ 12 & 3 \end{vmatrix} = 21 - 12 = 9$$

$$\Delta_2 = \begin{vmatrix} 3 & 7 \\ 5 & 12 \end{vmatrix} = 36 - 35 = 1$$

$$\therefore \quad x = \frac{\Delta_1}{\Delta} = \frac{9}{4}.$$

$$y = \frac{\Delta_2}{\Delta} = \frac{1}{4}$$

$\because$ solution is $x = \dfrac{9}{4}$, $y = \dfrac{1}{4}$.

Example 2. Solve the following system of equations by Cramer's rule :

$$x + 3y + 5z = 22$$
$$5x - 3y + 2z = 5$$
$$9x + 8y - 3z = 16.$$

Sol. The given equations are

$$x + 3y + 5z = 22$$
$$5x - 3y + 2z = 5$$
$$9x + 8y - 3z = 16$$

$$\text{Let } \Delta = \begin{vmatrix} 1 & 3 & 5 \\ 5 & -3 & 2 \\ 9 & 8 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} -3 & 2 \\ 8 & -3 \end{vmatrix} - 3\begin{vmatrix} 5 & 2 \\ 9 & -3 \end{vmatrix} + 5\begin{vmatrix} 5 & -3 \\ 9 & 8 \end{vmatrix}$$

$$= 1\,(9 - 16) - 3\,(-15 - 18) + 5\,(40 + 27)$$
$$= -7 + 99 + 335 = 427$$

$$\Delta_1 = \begin{vmatrix} 22 & 3 & 5 \\ 5 & -3 & 2 \\ 16 & 8 & -3 \end{vmatrix}$$

$$= 22\begin{vmatrix} -3 & 2 \\ 8 & -3 \end{vmatrix} - 3\begin{vmatrix} 5 & 2 \\ 16 & -3 \end{vmatrix} + 5\begin{vmatrix} 5 & -3 \\ 16 & 8 \end{vmatrix}$$

$$= 22\,(9 - 16) - 3\,(-15 - 32) + 5\,(40 + 48)$$
$$= -154 + 141 + 440 = 427$$

$$\Delta_2 = \begin{vmatrix} 1 & 22 & 5 \\ 5 & 5 & 2 \\ 9 & 16 & -3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 5 & 2 \\ 16 & -3 \end{vmatrix} - 22\begin{vmatrix} 5 & 2 \\ 9 & -3 \end{vmatrix} + 5\begin{vmatrix} 5 & 5 \\ 9 & 16 \end{vmatrix}$$

$$= 1\,(-15 - 32) - 22\,(-15 - 18) + 5\,(80 - 45) = -47 + 726 + 175 = 854$$

$$\Delta_3 = \begin{vmatrix} 1 & 3 & 22 \\ 5 & -3 & 5 \\ 9 & 8 & 16 \end{vmatrix}$$

$$= 1\begin{vmatrix} -3 & 5 \\ 8 & 16 \end{vmatrix} - 3\begin{vmatrix} 5 & 5 \\ 9 & 16 \end{vmatrix} + 22\begin{vmatrix} 5 & -3 \\ 9 & 8 \end{vmatrix}$$

$$= 1(-48-40) - 3(80-45) + 22(40+27) = -88 - 105 + 1474 = 1281$$

Now $x = \dfrac{\Delta_1}{\Delta} = \dfrac{427}{427} = 1$

$$y = \frac{\Delta_2}{\Delta} = \frac{854}{427} = 2$$

$$z = \frac{\Delta_3}{\Delta} = \frac{1281}{427} = 3$$

$\therefore$ solution is $x = 1, y = 2, z = 3$.

Example 3. Solve by Cramer's rule :

$$x + 4y - 2z = 3$$
$$3x + y + 5z = 7$$
$$2x + 3y + z = 5.$$

Sol. The given equations are

$$x + 4y - 2z = 3$$
$$3x + y + 5z = 7$$
$$2x + 3y + z = 5$$

Let $\Delta = \begin{vmatrix} 1 & 4 & -2 \\ 3 & 1 & 5 \\ 2 & 3 & 1 \end{vmatrix}$

$$= 1\begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix} - 4\begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 1(1-15) - 4(3-10) - 2(9-2) = -14 + 28 - 14 = 0$$

$$\Delta_1 = \begin{vmatrix} 3 & 4 & -2 \\ 7 & 1 & 5 \\ 5 & 3 & 1 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & 5 \\ 3 & 1 \end{vmatrix} - \begin{vmatrix} 7 & 5 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 7 & 1 \\ 5 & 3 \end{vmatrix}$$

$$= 3(1-15) - 4(7-25) - 2(21-5)$$

$$= -42 + 72 - 32 = -2$$

$\therefore$ given system of equations has no solution.

Example 4. Solve

$$2x - y + z = 4$$
$$x + 3y + 2z = 12$$
$$3x + 2y + 3z = 16$$

Sol. The given equations are

$$2x - y + z = 4 \quad \ldots(1)$$
$$x + 3y + 2z = 12 \quad \ldots(2)$$
$$3x + 2y + 3z = 16 \quad \ldots(3)$$

Let $\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 3 \end{vmatrix}$

$= 2(9-4) - (-1)(3-6) + 1(2-9)$

$= 10 - 3 - 7 = 0$

$$\Delta_1 = \begin{vmatrix} 4 & -1 & 1 \\ 12 & 3 & 2 \\ 16 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 4 & 5 & 2 \\ 3 & 5 & 3 \end{vmatrix} \text{ by } C_1 - 4C_3,\ C_2 + C_3$$

$$= 1 \begin{vmatrix} 4 & 5 \\ 4 & 5 \end{vmatrix} = 1(20-20) = 1 \times 0 = 0$$

$$\Delta_2 = \begin{vmatrix} 2 & 4 & 1 \\ 1 & 12 & 2 \\ 3 & 16 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -3 & 4 & 2 \\ -3 & 4 & 3 \end{vmatrix},$$

by $C_1 - 2C_3$, $C_2 - 4C_3$

$$= 1 \begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = 1(-12 + 12) = 1 \times 0 = 0$$

$$\Delta_3 = \begin{vmatrix} 2 & -1 & 4 \\ 1 & 3 & 12 \\ 3 & 2 & 16 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 7 & 3 & 24 \\ 7 & 2 & 24 \end{vmatrix},$$

by $C_1 + 2C_2$, $C_3 + 4C_2$

$$= -(-1) \begin{vmatrix} 7 & 24 \\ 7 & 24 \end{vmatrix} = 1(168 - 168) = 1 \times 0 = 0$$

$\therefore \quad \Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\therefore \quad$ given equations have infinite solutions.

(1) − 2 × (2) gives us

$$7y - 3z = -20, \qquad \therefore \quad 7y = 20 - 3z$$

$$\therefore \quad y = \frac{20 - 3z}{7}$$

$$\therefore \quad \text{from (2), } x + \frac{60 - 9z}{7} + 2z = 12$$

$$\therefore \quad x = 12 - \frac{60 - 9z}{7} - 2z = \frac{84 - 60 + 9z - 14z}{7}$$

$$\therefore \quad x = \frac{24 - 5z}{7}$$

Put $z = k$, $\therefore$ we have

$$x = \frac{24 - 5k}{7}, y = \frac{20 - 3k}{7}, z = k$$

For different values of k, these give the infinite solutions of given equations.

Example 5 . Solve the following system of equations :

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1, \quad \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2.$$

Sol. The given equations are

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \quad \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1, \quad \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Put $\frac{1}{x} = u$, $\frac{1}{y} = v$, $\frac{1}{z} = w$

$\therefore$ given equations become

$$2u + 3v + 10w = 4$$
$$4u - 6v + 5w = 1$$
$$6u + 9v - 20w = 2$$

$$\text{Let} \quad \Delta = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 & 10 \\ 0 & -12 & -15 \\ 0 & 0 & -50 \end{vmatrix}, \text{ by } R_2 - 2R_1, R_3, R_3 - 3R_1$$

$= (2)(-12)(-50)$ [Product of diagonal elements]

$= 1200$

$$\Delta_1 = \begin{vmatrix} 4 & 3 & 10 \\ 1 & -6 & 5 \\ 2 & 9 & -20 \end{vmatrix}$$

$$= 4\begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} - 3\begin{vmatrix} 1 & 5 \\ 2 & -20 \end{vmatrix} + 10\begin{vmatrix} 1 & -6 \\ 2 & 9 \end{vmatrix}$$

$$= 4\,(120 - 45) - 3\,(-20 - 10) + 10\,(9 + 12) = 300 + 90 + 210 = 600$$

$$\Delta_2 = \begin{vmatrix} 2 & 4 & 10 \\ 4 & 1 & 5 \\ 6 & 2 & -20 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 5 \\ 2 & -20 \end{vmatrix} - 4\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} + 10\begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix}$$

$$= 2\,(-20 - 10) - 4\,(-80 - 30) + 10\,(8 - 6) = 60 + 440 + 20 = 400$$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & 4 \\ 4 & -6 & 1 \\ 6 & 9 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} -6 & 1 \\ 9 & 2 \end{vmatrix} - 3\begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix} + 4\begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix}$$

$$= 2\,(-12 - 9) - 3\,(8 - 6) + 4\,(36 + 36) = -42 - 6 + 288 = 240$$

$$\therefore \quad u = \frac{\Delta_1}{\Delta} = \frac{600}{1200} = \frac{1}{2}, \qquad v = \frac{\Delta_2}{\Delta} = \frac{400}{1200} = \frac{1}{3},$$

$$w = \frac{\Delta_3}{\Delta} = \frac{240}{1200} = \frac{1}{5}$$

$$\therefore \quad x = \frac{1}{u} = 2, \qquad y = \frac{1}{v} = 3, \qquad z = \frac{1}{w} = 5$$

$\therefore$ solution is $x = 2,\ y = 3, z = 5$.

Example 6. Determine quadratic function defined by the equation $f(x) = a\,x^2 + b\,x + c$ if $f\,(0) = 6, f(2) = 11$ and $f(-3) = 6$.

Sol. Here $f\,(x) = a\,x^2 + b\,x + c$

Now $f(0) = 6$

$\Rightarrow \quad 0.a + 0.b + c = 6$ ...(1)

$f(2) = 11$

$\Rightarrow \quad 4a + 2b + c = 11$ ...(2)

$f(-3) = 6$

$\Rightarrow \quad 9a - 3b + c = 6$

Now we are to solve (1), (2), (3) for values a, b, c.

$$\text{Let } \Delta = \begin{vmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 9 & -3 & 1 \end{vmatrix} = 1 \begin{vmatrix} 4 & 2 \\ 9 & -3 \end{vmatrix} = -12 - 18 = -30$$

$$\Delta_1 = \begin{vmatrix} 6 & 0 & 1 \\ 11 & 2 & 1 \\ 6 & -3 & 1 \end{vmatrix} = 6(2+3) + 1(-33-12) = 30 - 45 = -15$$

$$\Delta_2 = \begin{vmatrix} 0 & 6 & 1 \\ 4 & 11 & 1 \\ 9 & 6 & 1 \end{vmatrix} = -6(4-9) + 1(24-99) = 30 - 75 = -45$$

$$\Delta_3 = \begin{vmatrix} 0 & 0 & 6 \\ 4 & 2 & 11 \\ 9 & -3 & 6 \end{vmatrix} = 6(-12-18) = 6(-30) = -180$$

$$a = \frac{\Delta_1}{\Delta} = \frac{-15}{-30} = \frac{1}{2}, \qquad b = \frac{\Delta_2}{\Delta} = \frac{-45}{-30} = \frac{3}{2},$$

$$c = \frac{\Delta_3}{\Delta} = \frac{-180}{-30} = 6$$

$\therefore$ we have, $\quad f(x) = \frac{1}{2}x^2 + \frac{3}{2}x + 6$

4.11 SOLUTION OF LINEAR EQUATION BY USING THE METHOD OF MATRIX INVERSE

4.11.1 System of linear equations

Consider a system of three non-homogeneous equations in three variables

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad \ldots(1)$$

These equations can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or $\quad AX = B$

$$\text{where } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Any set of values of x, y, z which satisfies the equations (1) is called a solution of the system (1).

Criterion of consistency

Ler the system of given equations be AX = B where A is a square .Then

(i) If | A | is not equal to 0, then the system is consistent and has a unique solution.

(ii) If | A | = 0 and (adj. A) B = O, then the system is consistent and has infinitely many solutions.

(ii) If | A | = 0 and (adj. A) B * O, then the system is consistent and has infinitely many solutions.

At this level, we are not in a position to prove these results. So, we take them as standard

results without proof.

Working rule for solving the linear equations.

$$a_1 x + b_1 y = c_1$$

and $$a_2 x + b_2 y = c_2$$

Step 1. Write the given equations in the form

$$AX = B$$

where $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Step 2. Evaluate | A |.

Two cases arise :

Case I. $|A| \neq 0$ *i.e.*, A^{-1} exists

$$AX = B$$

$$\Rightarrow \quad A^{-1}AX = A^{-1}B \quad \Rightarrow \quad IX = A^{-1}B \quad \Rightarrow \quad X = A^{-1}B$$

From the equality of two matrices, we get the values of x and y.

Case II. $|A| = 0$

Now two sub-cases arise :

Sub-case (*i*) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

In this case the given equation do not have any solution.

Sub-case (*ii*) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

System of homogeneous linear equations

Consider the equations

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$
$$a_3 x + b_3 y + c_3 z = 0$$

These equations can be written as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $\quad AX = O$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Here A is called coefficient matrix and the system of equations.

AX = O is called system of homogeneous linear equations.

Two Important Results:

Consider the system of equations AX = O

(i) If | A | is not equal to 0, then the system has only trivial solution i.e. $x = y = z = 0$

(ii) *(ii)* If | A | = 0, then the system has infinitely many solutions'

Illustrations:

Example 1. Use matrix method to solve the system of equations :

$$4x + 2y = 3$$
$$3x - 4y = 5.$$

Sol. The given equations are

$$4x + 2y = 3$$
$$3x - 4y = 5$$

These equations can be written as

$$\begin{bmatrix} 4 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

or $\quad AX = B$

where $A = \begin{bmatrix} 4 & 2 \\ 3 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$|A| = \begin{bmatrix} 4 & 2 \\ 3 & -4 \end{bmatrix} = -16 - 6 = -22 \neq 0$$

$\therefore \quad A^{-1}$ exists.

$$\text{adj. } A = \begin{bmatrix} -4 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -2 \\ -3 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj. } A}{|A|} = -\frac{1}{22} \begin{bmatrix} -4 & -2 \\ -3 & 4 \end{bmatrix}$$

Now $AX = B \Rightarrow X = A^{-1} B$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{22} \begin{bmatrix} -4 & -2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{22} \begin{bmatrix} -12 - 10 \\ -9 + 20 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{22} \begin{bmatrix} -22 \\ 11 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -\dfrac{1}{2} \end{bmatrix}$$

$$\Rightarrow \quad x = 1, y = -\frac{1}{2} \text{ is the solution.}$$

Example 2. Use matrix method to examine whether the system of equations

$$4x - 2y = 3$$
$$6x - 3y = 5$$

is consistent or not.

Sol. The given equations are

$$4x - 2y = 3 \qquad \ldots(1)$$
$$6x - 3y = 5 \qquad \ldots(2)$$

These equations can be written as

$$\begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

or $\quad AX = B$

where $A = \begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

$$|A| = \begin{vmatrix} 4 & -2 \\ 6 & -3 \end{vmatrix} = -12 + 12 = 0$$

$\therefore$ given equations either have no solution or have infinite solutions.

The equations (1) and (2) are inconsistent if

$$\frac{\text{coeff. of } x \text{ in (1)}}{\text{coeff. of } x \text{ in (2)}} = \frac{\text{coeff. of } y \text{ in (1)}}{\text{coeff. of } y \text{ in (2)}} \neq \frac{\text{constant term in (1)}}{\text{constant term in (2)}}$$

i.e., if $\dfrac{4}{6} = \dfrac{-2}{-3} \neq \dfrac{3}{5}$, which is so.

$\therefore$ given system of equations is inconsistent.

Example 3. Use matrix method to solve the system of equations

$$6x + 4y = 2$$
$$9x + 6y = 3$$

Sol. The given equations are

$$6x + 4y = 2 \qquad \ldots(1)$$

and $\quad 9x + 6y = 3 \qquad \ldots(2)$

These equations can be written as

$$\begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

or $\quad AX = B$

where $A = \begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 6 & 4 \\ 9 & 6 \end{vmatrix} = 36 - 36 = 0$$

$\therefore$ given equations have either no solution or have infinite solutions,

Now we have,

$$\frac{\text{coeff. of } x \text{ in (1)}}{\text{coeff. of } x \text{ in (2)}} = \frac{\text{coeff. of } y \text{ in (1)}}{\text{coeff. of } y \text{ in (2)}} = \frac{\text{constant term in (1)}}{\text{constant term in (2)}}$$

$$\left[\because \frac{6}{9} = \frac{4}{6} = \frac{2}{3}\right]$$

$\therefore$ given equations are consistent and have infinite solutions.

Giving different values to x, we get corresponding values of y from (1) or (2), and therefore, we get required infinite solutions.

Example 4. Use matrix method to solve the system of equations :

$$2x_1 - x_2 + x_3 = 4$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1 - 3x_2 - 2x_3 = 2$$

Sol. The given equations are

$$2x_1 - x_2 + x_3 = 4$$
$$x_1 + x_2 + x_3 = 1$$
$$x_1 - 3x_2 - 2x_3 = 2$$

These equations can be written as

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

or $\quad AX = B$

where $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & -2 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & -2 \end{vmatrix} = 2\begin{vmatrix} 1 & 1 \\ -3 & -2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}$$

$$= 2(-2+3) + 1(-2-1) + 1(-3-1)$$

$$= 2 - 3 - 4 = -5 \neq 0$$

$\therefore$ A^{-1} exists.

Co-factor of the elements of first row of $|A|$ are

$$\begin{vmatrix} 1 & 1 \\ -3 & -2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}$$

or $\quad -2+3, (-2-1), -3-1$ *i.e.* $1, 3, -4$ respectively

Co-factors of the elements of second row of $|A|$ are

$$-\begin{vmatrix} -1 & 1 \\ -3 & -2 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}, -\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix}$$

i.e. $-(2+3), -4-1, -(-6+1)$ *i.e.* $-5, -5, 5$ respectively.

Co-factors of the elements of third row of $|A|$ are

$$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$

i.e. $-1-1, -(2-1), 2+1$ *i.e.* $-2, -1, 3$ respectively.

$$\therefore \text{adj. } A = \begin{bmatrix} 1 & 3 & -4 \\ -5 & -5 & 5 \\ -2 & -1 & 3 \end{bmatrix}' = \begin{bmatrix} 1 & -5 & -2 \\ 3 & -5 & -1 \\ -4 & 5 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj. } A}{|A|} = -\frac{1}{5}\begin{bmatrix} 1 & -5 & -2 \\ 3 & -5 & -1 \\ -4 & 5 & 3 \end{bmatrix}$$

Now $\quad AX = B$

$\Rightarrow \quad X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 1 & -5 & -2 \\ 3 & -5 & -1 \\ -4 & 5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 4-5-4 \\ 12-5-2 \\ -16+5+6 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore \quad x_1 = 1,\ x_2 = -1,\ x_3 = 1.$

Example 5. Solve the following system of equations by matrix method :

$$3x + 4y + 7z = 14$$
$$2x - y + 3z = 4$$
$$x + 2y - 3z = 0$$

Sol. The given equations are

$$3x + 4y + 7z = 14$$
$$2x - y + 3z = 4$$
$$x + 2y - 3z = 0$$

These equations can be written as

$$\begin{bmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \\ 0 \end{bmatrix}$$

or $\quad AX = B$

where $A = \begin{bmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 14 \\ 4 \\ 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} + 7\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= 3(3-6) - 4(-6-3) + 7(4+1)$$

$$= -9 + 36 + 35 = 62 \neq 0$$

$\therefore \quad A^{-1}$ exists.

Co-factors of the elements of first row of $|A|$ are

$$\begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix}, -\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix}, \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix}$$

i.e., $-3, 9, 5$ respectively.

Co-factors of the elements of second row of | A | are

$$-\begin{vmatrix} 4 & 7 \\ 2 & -3 \end{vmatrix}, \begin{vmatrix} 3 & 7 \\ 1 & -3 \end{vmatrix}, -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$$

i.e., $26, -16, -2$ respectively.

Co-factors of the elements of third row of | A | are

$$\begin{vmatrix} 4 & 7 \\ -1 & 3 \end{vmatrix}, -\begin{vmatrix} 3 & 7 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix}$$

i.e., $19, 5 - 11$ respectively.

$\therefore$ adj.

$$A = \begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}' = \begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj. A}}{|A|} = \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}$$

Now $AX = B$

$\Rightarrow \quad X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}\begin{bmatrix} 14 \\ 4 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -42 + 104 + 0 \\ 126 - 64 + 0 \\ 70 - 8 + 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} 62 \\ 62 \\ 62 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad x = 1, y = 1, z = 1$ is the required solution.

Example 6. Compute A^{-1} for the following matrix

$$A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Hence solve the system of equations

$$-x + 2y + 5z = 2$$
$$2x - 3y + z = 15$$
$$-x + y + z = -3$$

Sol. Here $A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + 5\begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix}$$

$$= -(-3-1) - 2(2+1) + 5(2-3)$$

$$= 4 - 6 - 5 = -7 \neq 0$$

$\therefore \quad A^{-1}$ exists

Co-factors of the elements of the first row of $|A|$ are

$$\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix} \textit{ i.e } -4, -3, -1 \text{ respectively.}$$

Co-factors of the elements of the second row of $|A|$ are

$$-\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 5 \\ -1 & 1 \end{vmatrix}, -\begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} \textit{ i.e., } 3, 4, -1 \text{ respectively}$$

Co-factors of the elements of the third row of $|A|$ are

$$-\begin{vmatrix} 2 & 5 \\ -3 & 1 \end{vmatrix}, -\begin{vmatrix} -1 & 5 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} \textit{ i.e., } 17, 11, -1 \text{ respectively.}$$

$$\therefore \quad \text{adj. A} = \begin{bmatrix} -4 & -3 & -1 \\ 3 & 4 & -1 \\ 17 & 11 & -1 \end{bmatrix}' = \begin{bmatrix} -4 & 3 & 17 \\ -3 & 4 & 11 \\ -1 & -1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{adj.\,A}{|\,A\,|} = -\frac{1}{7}\begin{bmatrix} -4 & 3 & 17 \\ -3 & 4 & 11 \\ -1 & -1 & -1 \end{bmatrix}$$

The given equation are

$$-x + 2y + 5z = 2$$
$$2x - 3y + z = 15$$
$$-x + y + z = -3$$

These equation can be written as

$$\begin{bmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ -3 \end{bmatrix}$$

or $\quad AX = B$

where $A = \begin{bmatrix} -1 & 2 & 5 \\ 2 & -3 & 1 \\ -1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 15 \\ -3 \end{bmatrix}$

$\therefore \quad X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -4 & 3 & 17 \\ -3 & 4 & 11 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 15 \\ -3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -8 + 45 - 51 \\ -6 + 60 - 33 \\ -2 - 15 + 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -14 \\ 21 \\ -14 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$$

$\Rightarrow \quad x = 2, y = -3, z = 2$

Example 7. Which of the following equations are consistent and if consistent, find the solutions :

(*i*) $3x - y + 2z = 3$, $2x + y + 3z = 5$, $x - 2y - z = 1$

(*ii*) $x - y + z = 3$, $2x + y - z = 2$, $-x - 2y + 2z = 1$

Sol. (*i*) The given equations are

$$3x - y + 2z = 3$$
$$2x + y + 3z = 5$$
$$x - 2y - z = 1$$

These equations can be written as

$$\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

or $\quad AX = B$

where $\quad A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -2 & -1 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -1 & 2 \\ 2 & 1 & 3 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= 3(-1+6) - (-1)(-2-3) + 2(-4-1)$$
$$= 15 - 5 - 10 = 0$$

Co-factors of the elements of the first row of $|A|$ are

$$\begin{vmatrix} 1 & 3 \\ -2 & -1 \end{vmatrix}, \; -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix}, \; \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

or $\; -1 + 6, -(-2, -3), -4 - 1$

or $\; 5, 5 - 5$ respectively.

Co-factors of the elements of second row of $|A|$ are

$$-\begin{vmatrix} -1 & 2 \\ -2 & -1 \end{vmatrix}, \; \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}, \; -\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}$$

or $\quad -5, -5, 5$ respectively.

Co-factors of the elements of the third row of | A | are

$$\begin{vmatrix} -1 & 2 \\ 1 & 3 \end{vmatrix}, -\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}, \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}$$

or $\quad -5, -5, 5$ respectively.

$$\therefore \qquad \text{adj. A} = \begin{bmatrix} 5 & 5 & -5 \\ -5 & -5 & 5 \\ -5 & -5 & 5 \end{bmatrix}' = \begin{bmatrix} 5 & -5 & -5 \\ 5 & -5 & -5 \\ -5 & 5 & 5 \end{bmatrix}$$

$$(\text{adj. A})\,B = \begin{bmatrix} 5 & -5 & -5 \\ 5 & -5 & -5 \\ -5 & 5 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 15-25-5 \\ 15-25-5 \\ -15+25+5 \end{bmatrix} = \begin{bmatrix} -15 \\ -15 \\ 15 \end{bmatrix} \neq O$$

$\therefore$ given equations have no solution.

(*ii*) The given equations are

$$x - y + z = 3 \qquad \ldots(1)$$

$$2x + y - z = 2 \qquad \ldots(2)$$

$$-x - 2y + 2z = 1 \qquad \ldots(3)$$

These equation can be written as

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

or $\quad AX = B$

where $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & -3 \\ -1 & -3 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -3 \\ -3 & 3 \end{vmatrix} = 9 - 9 = 0$$

Co-factors of the elements of first row of | A | are

$$\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}, -\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix}$$

or $0, -3, -3$ respectively.

Co-factors of the elements of 2 nd row of | A | are

$$-\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}, -\begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix}$$

or $0, 3, 3$ respectively.

Co-factors of the elements of third row of | A | are

$$\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

or $0, 3, 3$ respectively.

$$\therefore \quad \text{adj. A} = \begin{bmatrix} 0 & -3 & -3 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 3 & 3 \\ -3 & 3 & 3 \end{bmatrix}$$

$$(\text{adj. A}) \quad B = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 3 & 3 \\ -3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0+0 \\ -9+6+3 \\ -9+6+3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$$

$\therefore$ given equations have infinite solutions.

Adding (1) and (2), we get, $3x = 5 \Rightarrow x = \dfrac{5}{3}$

Adding (1) and (3), we get, $-3y + 3z = 4$. Put $z = k$

$$\therefore \quad 3y = 3k - 4 \Rightarrow y = k - \frac{4}{3}$$

$$\therefore \quad \text{solutions are } x = \frac{5}{3}, y = k - \frac{4}{3}, z = k$$

where k is a parameter.

Example 8. Find non-trivial solution of the system:

$$x + y + z = 0$$
$$x - y - 5z = 0$$
$$x + 2y + 4z = 0$$

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Sol. The given system of equations is

$$x + y + z = 0$$
$$x - y - 5z = 0$$
$$x + 2y + 4z = 0$$

Let A be coefficient matrix.

$$\therefore \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -5 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\therefore \quad |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -5 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 1(-4+10) - 1(4+5) + 1(2+1)$$

$$= 6 - 9 + 3$$

$$= 0$$

$\therefore$ A is singular.

$\therefore$ given system of equations have infinitely many solutions.

Consider the first two equations

$$x + y + z = 0$$

$$x - y - 5z = 0$$

These equations can be written as

$$x + y = -z$$

$$x - y = 5z$$

Adding and subtracting these equations, we get,

$$x = 2z, \quad y = -3z$$

Put $\quad z = k$

$\therefore \quad x = 2k, y = -3k, z = k$

Putting these values of x, y, z in the third equation

$x + 2y + 4z = 0$, we get, $2k - 6k + 4k = 0$

or $\quad 0 = 0$, which is true

$\therefore$ all the equations are satisfied by $x = 2k$, $y = -3k$, $z = k$. These values of x, y, z give us infinitely many solutions for different real values of k

To conclude,

The inverse matrix method uses the inverse of a matrix to help solve a system of equations, such like the above Ax = b. By pre-multiplying both sides of this equation by A^{-1} gives:

$$A^{-1}(Ax) = A^{-1}b$$
$$(A^{-1}A)x = A^{-1}b$$

or alternatively

$$x = A^{-1}b$$

So by calculating the inverse of the matrix and multiplying this by the vector b we can find the solution to the system of equations directly. And from earlier we found that the inverse is given by

$$A^{-1} = \frac{adj(A)}{\det(A)}$$

From the above it is clear that the existence of a solution depends on the value of the determinant of A. There are three cases:

1. If the det(A) does not equal zero then solutions exist using $x = A^{-1}b$
2. If the det(A) is zero and b=0 then the solution will be not be unique or does not exist.
3. If the det(A) is zero and b=0 then the solution can be x = 0 but as with 2. is not unique or does not exist.

Similarly for three simultaneous equations we would have:

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

Written in matrix form would look like

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

and by rearranging we would get that the solution would look like

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

4.12 SUMMARY

- In business mathematics, one comes across mathematical objects, other than number, which are combined with one another through one or more compositions, whose laws are more or less identical with those of the compositions with numbers. 'Matrices' which constitute the subject of study in this unit.
- A matrix with *m* rows and *n* columns is called a matrix of order *m* x *n*.
- A square matrix is a matrix with an equal number of rows and columns. Since the number of rows and columns are the same, it is said to have order *n*.

- The main diagonal of a square matrix are the elements from the upper left to the lower right of the matrix.
- A row matrix is a matrix that has only one row.
- A column matrix is a matrix that has only one column.
- A matrix with only one row or one column is called a vector.

- Multiplication of two matrices A and B is possible if the number of columns of Prefactor is same as the number of rows of post factor.
- If A is $3\times \mathbf{4}$ **and B is** $\mathbf{4} \times \mathbf{3}$ **then AB** will be defined and will be of order $3\times \mathbf{3}$; **BA will also be defined and will be of order** $\mathbf{4} \times \mathbf{4}$.
- If A is 3 * 4 and B is also 3 *4 then neither AB or BA is defined.
- If A is 3*3 and B is 3 *2 AB is defined but not BA.
- The rank of a non-zero matrix is the largest order of any non-vanishing minor of the matrix.
- For matrix of higher order, the rank finding is difficult.
- Elementary transformation of matrix is a method of altering a matrix without changing its rank.
- Elementary transformation is called a row transformation or a column transformation according as it applies to rows or columns.
- Elementary Row Operations are operations that can be performed on a matrix that will produce a row-equivalent matrix. If the matrix is an augmented matrix, constructed from a system of linear equations, then the row-equivalent matrix will have the same solution set as the original matrix.
- Adjoint of any matrix A m*n is obtained by replacing every (i, j)th elements of A by the Co-factor of the (j,i)th element in A.

- A (adj. A) = IAI * I = (adj. A)a
- A-1 is inverse of matrix A if
- A-1A = AA-1 = I
- If [A] = 0, A-1 does not exist.
- If [A] $\neq \mathbf{0}$, $\boldsymbol{A}$ **is non singular matrix**.
- A n*n determinant is of order n.
- Sarus's rule is an alternative method of determining the determinant.
- AX = 0 (linera homogeneous form)
- AX= B (linear non homogeneous form)
- Rij, Ri(c) and Rij (k) are row operations.
- Cramer rule is used for solution of linear non homogeneous equations only if [A] $\neq \mathbf{0}$.

4.13 SELF ASSESSMENT EXERCISES

1. If a matrix A has 12 elements, what are the possible orders it can have? What if it has 7 elements?
2. If a matrix has 8 elements, what are the possible orders it can have?
3. Construct a 2 X 4 matrix whose elements are aij = 2i-3j.
4. The co-operative store of a particular school has 10 dozen physics books, 8 dozen chemistry books and 5 dozen mathematics books. Their selling prices are Rs. 8-30, Rs. 3-45 and Rs. 4-50 each respectively. Find the total amount the store will receive from selling all the items.
5. A factory produces three items A, B and C. Annual sales of the products are as given below :

City Products

Delhi:	5,000	1,000	20,000
Bombay:	6,000	10,000	8,000

If the unit sale price of the total products are Rs. 2-50, Re. 1-25, Re. 1 -50 respectively, find the total revenue in each city, with the help of matrix.

6. Show that the elements on the main diagonal of a skew symmetric matrix are all zero.
7. If a matrix A is symmetric as well as skew symmetric, then A = O.
8. If A be any square matrix, then show that A + A' is symmetric and A - A' is skew-symmetric.
9. A is symmetric; show that for any matrix B, B'AB is symmetric.
10. A and B are symmetric show that AB + BA is symmetric and AB - BA is skew-symmetric.

4.14 SUGGESTED READING

- Linear Algebra (Spectrum) Sharma Publications
- Raj, M. And Seth, S.S.: Elements of Matrices and determinants
- Mehta, B.C and Rachana G.M.K: Basic Maths for Commerce
- S. Narayan: A text Book of Matrices
- Sancheti, B.C and Kapoor V.K: Business Mathematics

SKILL DEVELOPMENT

STRUCTURE:

5.1 Introduction

5.2 Objective

5.3 Importance of Business Mathematics

5.4 Arithmetic Mean

5.5 Geometric Means

5.6 Relationship between A.M. and G.M.

5.7 Arithmetic mean, Geometric mean and Harmonic mean

5.8 Operation on Sets

5.9 Union and Intersection of Three Sets.

5.10 Concept of Compound Interest

5.11 Finding the Compound Interest

5.12 Matrix Multiplication

5.13 Discount on bill discounted

5.14 Gain or Loss Percent.

5.15 Selling Price calculation

5.16 Cramer’s Rule

5.17 Simple Interest

5.18 Cost Price

5.19 Original Rate and Reduced Rate

5.20 How higher purchase system is different from Instalment payment system

5.21 Reasons for varied EMI payments

5.1 Introduction

We discuss what we consider important in mathematics education: the mathematical literacy and the organisation of the mathematical content. The mathematical competencies that are needed can be categorised into three “levels” and the mathematical concepts into strands or “big ideas.” We then discuss the whole array of formats and tools that are available for classroom assessment. Feedback and scoring are discussed before finally discussing more practical realizations of such a framework into the classroom.

The aim of classroom assessment is to produce information that contributes to the teaching and learning process and assists in educational decision making, where decision makers include students, teachers, parents, and administrators. The aim of mathematics education is to help students become mathematically literate. This means that the individual can deal with the mathematics involved in real world problems (i.e. nature, society, culture—including mathematics) as needed for that individual’s current and future private life (as an intelligent citizen) and occupational life (future study or work) and that the individual understands and appreciates mathematics as a scientific discipline. The aim of a framework for classroom assessment in mathematics is to bring the aim of classroom assessment together with the aim of mathematics education in a seamless and coherent way, with optimal results for the teaching and learning process, and with concrete suggestions about how to carry out classroom assessment in the classroom situation.

5.2 Objective:

- To make students acquainted with the specimen for the classroom teaching pattern and internal assessment.

5.3 What is Importance of Business Mathematics in Management?

Mathematics is an important subject and knowledge of it enhances a person’s reasoning, problem-solving skills, and in general, the ability to think. Hence, it is important for understanding almost every subject whether science and technology, medicine, the economy, or business and finance. Mathematical tools such as the theory of chaos are used in mapping market trends and forecasting. Statistics and probability which are branches of mathematics are used in everyday business and economics. Mathematics is also an important part of accounting, and many accountancy companies prefer graduates with joint degrees with mathematics rather than just an accountancy qualification. Financial Mathematics and Business Mathematics are two important branches of mathematics in today's world and these are direct application of mathematics to business and economics.
Examples of applied maths such as probability theory and management science, such as queuing theory, time-series analysis, linear programming all are vital maths for business.Importance of Business Mathematics in Management is sometimes called commercial math or consumer math, is a group of practical subjects used in commerce and everyday life. Business Mathematics in Management is the differences in coursework from a business mathematics course and a regular mathematics course would be calculus. In a regular calculus course, students would study trigonometric functions. Business calculus would not study trigonometric functions because it would be time-consuming and useless to most business students, except perhaps economics majors.
Economics majors who plan to continue economics in graduate school are strongly encouraged to take regular calculus instead of business calculus, as well as linear algebra and other advanced math courses, especially Real Analysis.
On the other way, meaning of business mathematics includes mathematics courses taken at an undergraduate level by business students. These courses are

slightly less difficult and do not always go into the same depth as other mathematics courses for people majoring in mathematics or science fields. The two most common math courses taken in this form are Business Calculus and Business Statistics. Examples used for problems in these courses are usually real-life problems from the business world.
A U.S. business math course might include a review of elementary arithmetic, including fractions, decimals, and percentages. Elementary algebra is often included as well, in the context of solving practical business problems. The practical applications typically include checking accounts, price discounts, mark ups and mark downs, payroll calculations, simple and compound interest, consumer and business credit, and mortgages.

In academia, "Business Mathematics in Management system" includes mathematics courses taken a tan undergraduate level by business students. These courses are slightly less difficult and do not always go into the same depths other mathematics courses for people major in mathematics or science fields. The two most common math courses taken in this form are Business Calculus and Business Statistics. Examples used for problems in these courses are usually real-life problems from the business world.

Mathematics is used in most aspects of daily life. Many of the top jobs such as business consultants, computer consultants, airline pilots, company directors and a host of others require a solid understanding of basic mathematics, and in some cases require a quite detailed knowledge of mathematics. It also play important role in business, like Business mathematics by commercial enterprises to record and manage business operations. Mathematics typically used in commerce includes elementary arithmetic, such as fractions, decimals, and percentages, elementary algebra, statistics and probability.

An example of the differences in course work from a business mathematics course and a regular mathematics course would be calculus. In a regular calculus course, students would study trigonometric functions. Business calculus would not study trigonometric functions because it would be time-consuming and useless to most business students, except perhaps economics majors. Economics majors who plan to continue economics in graduate school are strongly encouraged to take regular calculus instead of business calculus, as well as linear algebra and other advanced math courses. Other subjects typically covered in a business mathematics curriculum include Matrix algebra linear programming.

5.4 Insert Five Arithmetic Means between 8 and 26.

Let A_1, A_2, A_3, A_4 and A_5 be five arithmetic means between 8 and 26.

Therefore, 8, A_1, A_2, A_3, A_4, A_5, 26 are in A. P. with $a = 8$, $b = 26$, $n = 7$

We have $26 = 8 + (7 - 1)\, d$

$\therefore d = 3$

$\therefore$ A1 $= a + d = 8 + 3 = 11$,

A2 $= a + 2d = 8 + 2 \times 3 = 14$

A3 $= a + 3d = 17$, A4 $= a + 4d = 20$

A5 $= a + 5d = 23$

Hence, the five arithmetic means between 8 and 26 are 11, 14, 17, 20 and 23.

5.5 Insert Three Geometric Means between 1 and 256.

Let G_1, G_2, G_3, be the three geometric means between 1 and 256.

Then 1, G_1, G_2, G_3, 256 are in G. P.

If r be the common ratio, then $a_5 = 256$

i.e, $ar^4 = 256 \Rightarrow 1.\ r^4 = 256$

or, $r^2 = 16$

or, $r = \pm 4$

When $r = 4$, $G_1 = 1.\ 4 = 4$, G2 $= 1.\ (4)^2 = 16$ and G3 $= 1.\ (4)^3 = 64$

When $r = -4$, $G_1 = -4$, G2 $= (1)\ (-4)^2 = 16$ and G3 $= (1)\ (-4)^3 = -64$

$\therefore$G.M. between 1 and 256 are 4, 16, 64, or, – 4, 16, –64.

5.6 The Arithmetic Mean between Two Numbers is 34 and their Geometric Mean is 16. Find The Numbers.

Let the numbers be a *and* b.

Since A.M. between a and b is 34,

$a + b/2 = 34$, or, $a + b = 68$...................................... (1)

Since G. M. between a and b is 16,

$\sqrt{ab} = 16$ or, $ab = 256$

we know that $(a – b)^2 = (a + b)^2 – 4$ ab..........................(2)

$= (68)^2 – 4 \times 256$

$= 4624 – 1024 = 3600$

$\therefore a – b = 3600 = 60$.. (3)

Adding (1) and (3), we get, $2a = 128$

$\therefore a = 64$

Subtracting (3) from (1), are get

$2b = 8$ or, $b = 4$

$\therefore$ Required numbers are 64 and 4.

5.7 If the A.M between two numbers is 1, prove that their H.M is the square of their G.M.

Arithmetic mean between two numbers is 1.

ie. = (a+b)/2 = 1

⇨ a +b=2

Now H.M = $2ab/a + b$= ab

$\therefore$ $(G.M)^2 = \sqrt{ab}$

$(G.M)^2 =$ ab

5.8 If A = {1, 3, 5}, B = {3, 5, 6} and C = {1, 3, 7}

(i) Verify that A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

(ii) Verify A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

(i) A∪(B∩C)=(A∪B)∩(A∪C)

L.H.S.=A∪(B∩C)

B∩C={3}

$A \cup (B \cap C) = \{1,3,5\} \cup \{3\} = \{1,3,5\}$ ……………….. (1)

R.H.S.$=(A \cup B) \cap (A \cup C)$

$A \cup B = \{1,3,5,6\}$
$A \cup C = \{1,3,5,7\}$
$(A \cup B) \cap (A \cup C) = \{1, 3, 5, 6\} \cap \{1, 3, 5, 7\} = \{1, 3, 5\}$
……………….. (2)

From (1) and (2), we conclude that;

$A \cup (B \cap C) = A \cup B \cap (A \cup C)$ [*verified*]

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

L.H.S. $= A \cap (B \cup C)$

$B \cup C = \{1, 3, 5, 6, 7\}$

$A \cap (B \cup C) = \{1, 3, 5\} \cap \{1, 3, 5, 6, 7\} = \{1, 3, 5\}$ ……………….. (1)
R.H.S. $= (A \cap B) \cup (A \cap C)$

$A \cap B = \{3, 5\}$

$A \cap C = \{1, 3\}$

$(A \cap B) \cup (A \cap C) = \{3, 5\} \cup \{1, 3\} = \{1, 3, 5\}$ ……………….. (2)

From (1) and (2), we conclude that;

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [*verified*]

5.9 Let $A = \{a, b, d, e\}$, $B = \{b, c, e, f\}$ and $C = \{d, e, f, g\}$

(i) Verify $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) Verify $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

L.H.S. = $A \cap (B \cup C)$

$B \cup C = \{b, c, d, e, f, g\}$

$A \cap (B \cup C) = \{b, d, e\}$ ……………….. (1)

R.H.S. = $(A \cap B) \cup (A \cap C)$

$A \cap B = \{b, e\}$

$A \cap C = \{d, e\}$

$(A \cap B) \cup (A \cap C) = \{b, d, e\}$ ……………….. (2)

From (1) and (2), we conclude that;

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [verified]

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

L.H.S. = $A \cup (B \cap C)$

$B \cap C = \{e, f\}$

$A \cup (B \cap C) = \{a, b, d, e, f\}$ ……………….. (1)

R.H.S. = $(A \cup B) \cap (A \cup C)$

$A \cup B = \{a, b, c, d, e, f\}$

$A \cup C = \{a, b, d, e, f, g\}$

$(A \cup B) \cap (A \cup C) = \{a, b, d, e, f\}$ ……………….. (2)

From (1) and (2), we conclude that;

$A \cup (B \cap C) = A \cup B \cap (A \cup C)$ [verified]

5.10 Explain the Concept of Compound Interest With Example.

Suppose that one cent had been invested 2015 years ago at a constant annual interest rate of 2%. After the first year, this interest rate was applied to the initial principal of one cent and the capital grew to 1.02 cent. In the second year, the interest earned was again 2%. However, from that time onwards, it was not applied to the principal only but to the compound capital value (i.e., 1.02 cent). Thus, after the second year, the capital increased to 1.02×1.02 cent. After the third year, the capital grew to 1.02^3 cent. After 2015 years, the capital has eventually grown to 1.02^{2015} cent, which is roughly equal to 2.13×10^{17} cent or, more precisely, 213,474,546,813,926,768.7 cent.

Compare this figure to a similar investment using simple interest rather than compound interest. Suppose again that 1 cent is invested for a period of 2015 years at a constant annual interest rate of 2%. In this case, after 2015 years, the final capital is only 41.3 cent. This comparison highlights the effect of compounding, especially for long-term investments.

5.11 What will Rs.1500 amount to in three years if it is invested in 20% p.a. compound interest, interest being compounded annually?

The usual way to find the compound interest is given by the formula A = $P(1+r/100)^t$

In this formula, A is the amount at the end of the period of investment;

P is the principal that is invested;

r is the rate of interest in % p.a

And t is the number of years for which the principal has been invested.

Let us look at another alternative.

What happens in compound interest?

Interest is paid on interest.

In the first year, interest is paid only on the principal. That is very similar to simple interest.

However, from the second year onwards things change. In the second year, you pay interest on the principal and also interest on interest.

Therefore, the Amount at the end of 2^{nd} year in compound interest can be computed as follows:

1 × Principal + 2× Simple interest on principal + 1 × interest on interest.

Similarly, if you were to find the Amount at the end of 3 years in compound interest use the following method

1×Principal + 3 ×Simple interest on principal + 3 × interest on interest + 1 ×interest on interest on interest

Let us see how it works in our example.

The principal is Rs.1500. The rate of interest is 20%. Therefore, the simple interest on principal is 20% of 1500 = Rs.300

The interest on interest = 20% interest on the interest of Rs.300 = 20% of Rs.300 = Rs.60.

Interest on interest on interest = 20% of Rs.60 = Rs.12.

Now add all these

Amount at the end of 3 years = 1×Principal + 3 × Simple interest on principal + 3 ×interest on interest + 1 × interest on interest on interest

= 1500 + 3×300 + 3×60 + 1×12 = 1500 +900 + 1800 +12 = 2592.

You will get the same answer if you had used the formula. However, the calculation in this case was far easier than using the formula.

5.12 A radio manufacturing company produces three models of radios say A, B and C. There is an export order of 500 for model A, 1000 for model B, and 200 for model C. The material and labour (in appropriate units) needed to produce each model is given by the following table:

$$\begin{array}{ll} & \begin{array}{cc}\textbf{Material} & \textbf{labour}\end{array} \\ \text{Model A} & \begin{pmatrix} 10 & 20 \\ 8 & 5 \\ 12 & 9 \end{pmatrix} \end{array}$$

Model B

Model C

Use matrix multiplication to compute the total amount of material and labour needed to fill the entire export order.

Let P denote the matrix expressing material and labour corresponding to the models A, B,C.

Material Labour

$$\text{Then, } P = \begin{pmatrix} 10 & \cdots & 20 \\ 8 & \ddots & 5 \\ 12 & \cdots & 9 \end{pmatrix} \text{Model A, Model B, Model C}$$

Let E denote matrix expressing the number of units ordered for export in respect of models A, B, C. Then

A B C

E = (500 1000 200)

∴Total amount of material and labour = E x P

= (5000 + 8000 + 2400 10000 + 5000 + 1800)

Material Labour

= (15,400 16,800)

5.13 A drawer has a bill for $10,000. He discounted this bill with his bank two months before its due date at 15% p.a. rate of discount. calculate discount.

Discount will be calculated as the follow:

$1,000 \times 15/100 \times 2/12 = 250$

Thus, the drawer will receive a cash worth $9,750 and will bear a loss of $250.

The bank will keep this bill in possession till the due date. On maturity (due date) the bank will present the bill to the acceptor and will receive cash from him (if the bill is honoured). In case, the acceptor does not make the payment to the bank, then the drawer on any person who has discounted the bill have to take this liability and will pay cash to the bank. Until the bill is honoured on the due date, there is always a chance that the drawer will become liable on the bill. This is called a contingent liability - a liability that will only arise if a certain event occurs - the acceptor does not honour the bill.

5.14 A shopkeeper buys an article for Rs. 360 and sells it for Rs. 270. Find his gain or loss percent.

Here C.P. = Rs. 360, and S.P. = Rs. 270
Since C.P. > S.P., $\therefore$ there is a loss.
Loss = C.P. – S.P. = Rs (360 – 270) = Rs. 90
Loss % = Loss/ C.P $\times 100$

$= 90/360 \times 100 = 25\%$

5.15 By selling a book for 258, a publisher gains 20%. For how much should he sell it to gain 30%?

Here, S.P. = Rs. 258
Profit = 20%

C.P. = SP*100/100+ Profit%

Now, if Profit = 30% and C.P. = Rs. 215, then,

S.P. = $\frac{\text{C.P}\times(100 + \text{Profit\%})}{100} = 279.50$

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5.16 Solve the system using Cramer's Rule:

x − 3y = 6; 2x + 3y = 3

First we determine the values we will need for Cramer's Rule:

$a_1 = 1$ $b_1 = -3$ $c_1 = 6$

$a_2 = 2$ $b_2 = 3$ $c_2 = 3$

$x= \begin{vmatrix} 6 & -3 \\ 3 & 3 \end{vmatrix} / \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 18 + 9/3 + 6 = 3$

$y= \begin{vmatrix} 1 & 6 \\ 2 & 3 \end{vmatrix} / \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 3 - \frac{12}{3} + 6 = -1$

so the solution is (3, -1).

5.17 A sum of money at simple interest amounts to Rs. 815 in 3 years and to Rs. 854 in 4 years. The sum is:

S.I. for 1 year = Rs. (854 - 815) = Rs. 39.

S.I. for 3 years = Rs.(39 x 3) = Rs. 117.

Principal = Rs. (815 - 117) = Rs. 698.

5.18 Henry sold a bicycle at 8% gain. Had it been sold for Rs. 75 more, the gain would have been 14%. Find the cost price of the bicycle.

Let the cost price of the bicycle be Rs. x.

SP of the bicycle at 8% gain =Rs. $[\{(100 + \text{gain }\%) /100\} \times CP]$

$= \text{Rs. } [\{(100 + 8)/100\} \times x]$

$= \text{Rs. } \{(108/100) \times x\}$

$= \text{Rs. } (27x/25)$

SP of the bicycle at 14% gain = Rs. $[\{(100 + 14)/100\} \times x]$

$= \text{Rs. } \{(114/100) \times x\}$

$= \text{Rs. } (57\,x/50)$

Therefore, $(57\,x /50) - (27\,x/25) = 75$

$\Leftrightarrow (57\,x - 54\,x)/50 = 75$

$\Leftrightarrow 3\,x = (50 \times 75)$

$\Leftrightarrow x = (50 \times 25)$

$\Leftrightarrow x = 1250$

Hence the CP of the bicycle is Rs. 1250.

5.19 A reduction of 20% in the price of sugar enables Mrs. Jones to buy an extra 5 kg of it for Rs. 320.

Find:
(i) the original rate, and

(ii) the reduced rate per kg.

Let the original rate be Rs. x per kg.

Reduced rate = (80% of Rs. x) per kg

= Rs. ($x \times 80/100$) per kg

Quantity of sugar for Rs. 320 at original rate = 320/x kg

Quantity of sugar for Rs. 320 at the new rate = 320/(4x/5) kg

= (320×5)/4x kg

= 400/x kg.

Therefore, (400/x) - (320/x) = 5

$\Leftrightarrow$ 5x = (400 - 300)

$\Leftrightarrow$ 5x = 80

$\Leftrightarrow$ x = 16

(i) **Original rate** = Rs. 16 per kg

(ii) **Reduced rate** = ($4/5 \times 16$) per kg

= Rs. 64/5 per kg

= Rs. 12.80 per kg.

5.20 Difference between Hire-purchase system and Instalment payment system

Instalment Payment System is system of purchase and sale of goods in which title of goods is immediately transferred to the purchaser at the time of sale of goods and the sale price of the goods is paid in instalments. In the event of default in payment of any instalment, the seller has no right to take back goods from the possession of the purchaser. He can file a suit for the recovery of the outstanding balance of the price of goods sold. The followings are the differences between Hire-purchase system and Instalment payment system:

- In Hire-purchase system, the transfer of ownership takes place after the payment of all instalments while in case of Instalment payment system, the ownership is transferred immediately at the time of agreement.
- In Hire-purchase system, the hire-purchase agreement is like a contract of hire though later on it may become a purchase after the payment of last instalment while in Instalment payment system, the agreement is like a contract of credit purchase.
- In case of default in payment, in Hire-purchase system the vendor has a right to back goods from the possession of the hire- purchase while in case of Instalment payment system, the vendor has no right to take back the goods from the possession of the purchaser; he can simply sue for the balance due.
- In Hire-purchase system, if the purchaser sells the goods to a third party before the payment of last instalment, the third party does not get a better title on the goods purchased. But in case of Instalment payment system, the third party gets a better title on the goods purchased.
- In Hire-purchase system the provisions of the Hire-purchase Act apply to the transaction while in case of Instalment payment system, the provisions of Sale of Goods Act apply to the transaction.

5.21 Reasons for varied EMI payments

There are three factors which governs the EMI payments, for e.g. the EMI payments are directly proportional to loan amount and interest rates and are inversely proportional to the tenure of loan. The higher the loan amount or interest rate, the higher is the EMI payments and vice versa.

In case of tenure of loan, though the amount of total interest to be paid increases with the increase in tenure, the EMI payments decrease with the increase in tenure.

The other major factor which determines the EMI payments is the type of interest on the loan. In case of fixed rate loans, the EMI payments remain constant during the tenure. In case of floating rate loans, the interest rates vary based on the prevailing market rates. Hence, the EMI payments also vary whenever there is a change in the base rates.

The other factor which effects the EMI payments is the pre-closure or partial payments made towards the loan. Any partial payments made towards the loan are deducted from the principal amount of the loan. This results in reduction of total interest that is to be paid.

“Generally an individual who is making a partial payment will be given an option to keep the tenure constant or keep the EMI constant. If one opts for keeping the tenure constant, the monthly EMI payments will be reduced. Similarly, if one opts for keeping the EMI constant, the tenure of the loan will be reduced,”